(1) (20 pts) Note that $f(r) = \int_0^{\pi/2} x^r \sin x\, dx$.

(a) (14 pts) Show that

$$\lim_{r\to\infty} r\left(\frac{2}{\pi}\right)^{r+1} f(r) = 1.$$

(b) (6 pts) Find a real number c and a positive number L for which

$$\lim_{r\to\infty} \frac{r^c \int_0^{\pi/2} x^r \sin x\, dx}{\int_0^{\pi/2} x^r \cos x\, dx} = L.$$

You can use (a) as a fact to answer (b).

(2) (20%) For what pairs (a, b) of positive real numbers does the improper integral

$$\int_b^\infty \left(\sqrt{\sqrt{x+a}-\sqrt{x}} - \sqrt{\sqrt{x}-\sqrt{x-b}}\right) dx$$

converge? Justify your answer.

(3) (20 pts) Evaluate the integral

$$\int\int_D \frac{ky \sin kr_1 \sin kr_2}{r_1 r_2} dxdy,$$

where k is a constant, $r_1 = [(x+1)^2 + y^2]^{1/2}$, $r_2 = [(x-1)^2 + y^2]^{1/2}$, and D is the half of the ellipse $[(x+1)^2 + y^2] + [(x-1)^2 + y^2] \leq 4$ which is in $y > 0$.

(4) (20 pts) Find the maximum and minimum values of the function $f(x, y, z) = x$ over the curve of intersection of the plane $z = x + y$ and the ellipsoid $x^2 + 2y^2 + 2z^2 = 8$.

(5) (20 pts) Find the area of the cap cut from the hemisphere $x^2 + y^2 + z^2 = 2$, $z \geq 0$, by the cylinder $x^2 + y^2 = 1$.

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