1. (20%) Label the following statements as being true or false. *(No explanation is needed. Each correct answer gets 2% and each wrong answer gets 0%)*:

   (a) Let $\mathcal{S}$ be a nonempty subset of $\mathcal{R}^n$. Then $(\mathcal{S}^\perp)^\perp = \text{Span } \mathcal{S}$.

   (b) Let $\mathcal{S}_1$ be a linearly independent subset of $\mathcal{R}^n$ and $\mathcal{S}_2$ be a generating set for $\mathcal{R}^n$. Then $\mathcal{S}_1$ cannot have more vectors than $\mathcal{S}_2$.

   (c) Let $A$ be $m \times n$ and let $P_W$ be the orthogonal projection matrix for Col $A$. Then $A\mathbf{x} = P_W\mathbf{b}$ is consistent for each $\mathbf{b} \in \mathcal{R}^m$.

   (d) Let $A_1$ and $A_2$ be $m \times n$ matrices. If $A_1\mathbf{x} = \mathbf{b}_1$ and $A_2\mathbf{x} = \mathbf{b}_2$ are consistent, then $\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix}$ is consistent.

   (e) Let $V$ be a finite dimensional inner product space and let $\mathcal{B}$ be a basis for $V$. Then $\langle f, g \rangle = [f]_\mathcal{B} \cdot [g]_\mathcal{B}$, for any $f, g \in V$.

   (f) There exists a $3 \times 3$ diagonalizable matrix whose characteristic polynomial is given by $t^3 - t^2 - 2t$.

   (g) Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for $\mathcal{R}^n$. Let $A$ be $n \times n$. If $\|A\mathbf{v}_i\| = \|\mathbf{v}_i\|$, for $i = 1, 2, \ldots, n$, then $A$ is orthogonal.

   (h) Let $T : \mathcal{R}^{99} \to \mathcal{R}^{100}$ be linear. Then there exist a pair of distinct vectors $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{R}^{99}$ such that $T(\mathbf{v}_1) = T(\mathbf{v}_2)$.

   (i) Let $Q$ and $A$ be $m \times m$ and $m \times n$ matrices respectively, and $\mathbf{b} \in \mathcal{R}^m$. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be solution sets to $A\mathbf{x} = \mathbf{b}$ and $QA\mathbf{x} = Q\mathbf{b}$ respectively. Then $\mathcal{S}_1$ is a subspace of $\mathcal{S}_2$.

   (j) Let $W_1$ and $W_2$ be subspaces of $\mathcal{R}^n$. If $\dim W_1 + \dim W_2 > n$, then $W_1 \cap W_2$ is a nonzero subspace of $\mathcal{R}^n$.

2. Let $V$ be the set of all $2 \times 2$ symmetric matrices and $B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. Let $T$ be a linear operator on $V$ defined by $T(A) = B^TAB$, for each $A \in V$.

   (a) (3%) Find a basis for $V$.

   (b) (12%) Find the eigenvalues of $T$.

3. (15%) Let $\mathcal{P}_2$ be the set of all polynomials with degree less than equal to 2. For any $p_1(x), p_2(x) \in \mathcal{P}_2$, their inner product is defined by

$$\langle p_1(x), p_2(x) \rangle = \int_{-1}^{1} p_1(x)p_2(x)dx.$$

   Let $W = \{1, x\}$ and $p(x) = x^2$. Find the unique polynomials $q(x) \in W$ and $r(x) \in W^\perp$ such that $p(x) = q(x) + r(x)$.

4. You have two coins. Coin A comes up heads with probability 1/4. Coin B comes up heads with probability 1/2. You choose one of these coins randomly and then you flip it 3 times.

   (a) (4%) What is the probability that you observe at least 2 heads?

   (b) (4%) If you observe 3 tails, what is the probability that you have chosen Coin A?

5. The random variables $X$ and $Y$ have the joint probability density function

$$f_{X,Y}(x,y) = C \times \exp\left(-\frac{4x^2+y^2}{8}\right),$$

   where $-\infty < x < \infty$ and $-\infty < y < \infty$.

   (a) (4%) Find the constant $C$.

   (b) (4%) Find the conditional probability density function $f_{X|Y}(x|y)$.

   (c) (8%) Let $U = X + 2Y$ and $V = X - Y$. What is the correlation coefficient of $U$ and $V$?

   (d) (8%) Let $Z = 3X + 2Y$. Derive the moment generating function $\phi_Z(s) = \mathrm{E}[e^{sZ}]$. You have to give the derivation, not just the answer.

6. You make $N$ phone calls. The length of the $i$th phone call, denoted by $T_i$ in minutes, is an exponential random variable with expected value $1/\lambda$, where the parameter $\lambda > 0$. The random variables $T_1, T_2, \ldots, T_N$ are independent and identically distributed. For any fraction of a minute at the end of a call, the phone company charges for a full minute. In other words, the phone company calculates its charge based on $W_i = \lceil T_i \rceil$ for the $i$th phone call. Here the ceiling function of a real number $x$, denoted by $\lceil x \rceil$, is the least integer number greater than or equal to $x$. For instance, $\lceil 3.1 \rceil = 4$ and $\lceil 3 \rceil = 3$.

   (a) (3%) What is the probability density function of $T_1$?

   (b) (7%) Derive the probability mass function of $W_1$. You have to give the derivation, not just the answer.

   (c) (8%) Let $L = \sum_{i=1}^{N} W_i$. Derive the probability mass function of $L$. You have to give the derivation, not just the answer.

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