Problem 1 (20%)

True or False. Two points for each question. No reasons needed in problem 1.

(A) A 3 x 3 invertible matrix may have -1, 0, and 1 as its eigenvalues.

(B) A 4 x 4 real matrix cannot have -1, $1 + i$, and $2 + i$ as three of its eigenvalues.

(C) If A and B are both n x n invertible matrices, then both AB and A + B are invertible.

(D) The set of solutions $y(x)$ to $x^2y'' + 3xy' - 3y = 0$ for $x > 0$ forms a subspace of dimension 2.

(E) The set of n x n anti-symmetric matrices is a subspace of dimension $n(n - 1)/2$.

(F) Laplace transform and Fourier transform are linear transformations.

(G) A column vector $(-1, 1)^t$ can be a generalized eigenvector of a 2 x 2 symmetric matrix.

(H) If the characteristic polynomial of a matrix is $\lambda^4 + 4\lambda^3 + 3\lambda^2 - 4\lambda - 4$, the matrix is both invertible and diagonalizable.

(I) Denote O_{3x3} as a 3 x 3 zero matrix and $A = A_{3x3}$ as another 3 x 3 matrix. Then
$\det(A_{3x3}) + \det(-A_{3x3}) = \det(A_{3x3} - A_{3x3}) = \det(O_{3x3}) = 0$ and $A_{3x3} - A_{3x3} = O_{3x3}$.

(J) Let B, C and D be three singular square matrices such that D = B + C. Then det(D) = det(B + C) = det(B) + det(C).

Problem 2 (13%)

For $x > 0$, solve $x^2y'' - 2xy' + 2y = x + x^3(\ln x)^2 + x^3e^x$

Problem 3 (7%)

Suppose $y(x) = c_1e^x\cos 2x + c_2e^x\sin 2x + c_3e^x + c_4e^{2x} + \cosh x$, where c_1, c_2, c_3, and c_4 are constants, is the solution to an ODE of the form:

$$\sum_{k=0}^{n} a_k \frac{d^k y(x)}{dx^k} = f(x)$$

where a_k are constants. Construct such an ODE with the lowest possible order.

Problem 4 (10%)

Let $\vec{V}$ be a solenoidal vector field (meaning that its divergence is zero) described by a stream function:

$$\psi(x,y) = -\frac{\Gamma}{4\pi}\ln(x^2 + y^2)$$

Is $\vec{V}$ irrotational everywhere? What is the circulation $\oint_C \vec{V}\cdot d\vec{r}$ for any arbitrary simple closed curve C enclosing the origin $(x, y) = (0, 0)$

Problem 5 (10%)

Consider a differential equation $y'' - xy = 0$, which can be solved by the method of series solution to give a general solution in the form of $y(x) = af(x) + bg(x)$ with two linearly independent base functions and two coefficients to be determined by boundary conditions.

$$f(x) = 1 + \sum_{k=1}^{\infty} \frac{x^{3k}}{(3k)(3k-1)(3k-3)(3k-4)\cdots 3\cdot 2}$$

$$g(x) = x + \sum_{k=1}^{\infty} \frac{x^{3k+1}}{(3k+1)(3k)(3k-2)(3k-3)\cdots 4\cdot 3}$$

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Can we solve it as an eigenvalue (Sturm-Liouville) problem $\mathcal{L}\{y\} = \lambda y$ in terms of a linear operator $\mathcal{L}$ and an eigenvalue λ? If yes, determine the operator and the corresponding base solutions. If not, give reasons.

Problem 6 (10%)

Solve the wave equation for one finite string that is fixed at $x = 0$, set free at $x = L$, and released stationary in the form of $f(x)$.

Problem 7 (8%)

Select the correct answer for the integral $\int_{-\infty}^{\infty} \frac{cosx}{\pi^2-4x^2} dx$. No calculation procedures needed in problem 7.

(A) π/2
(B) - π/2
(C) 1/2
(D) -1/2
(E) 0

Problem 8 (10%)

Expand the function $f(x) = x$ in the interval $-1 < x < 1$ in the Legendre series expansion of the form $f(x) = \sum_{k=0}^{\infty} A_k P_k(x)$, where the Legendre polynomials $P_k(x)$ is defined as:

$$P_k(x) = \frac{(2k-1)(2k-3)\cdots\cdots 1}{k!}\left[x^k - \frac{k(k-1)}{2(2k-1)}x^{k-2} + \frac{k(k-1)(k-2)(k-3)}{2\cdot 4(2k-1)(2k-3)}x^{k-4} - \cdots\cdots\right]$$

Write the first five terms of the expansion at least.

Problem 9 (12%)

If a function $u(x, t)$ is said to be piecewise continuous in the interval $(0, l)$, k is an integer, and $t > 0$, then $\widetilde{U_s}(k,t)$ is called the finite Fourier sine transform of $u(x, t)$ and defined as:

$$\widetilde{U_s}(k,t) = \int_0^l u(x,t) sin\frac{k\pi x}{l} dx$$

$u(x, t)$ can also be expressed in terms of $\widetilde{U_s}(k,t)$ as:

$$u(x,t) = \frac{2}{l}\sum_{k=1}^{\infty} \widetilde{U_s}(k,t) sin\frac{k\pi x}{l}$$

The finite Fourier cosine transform of $u(x, t)$ can be defined in a similar way as $\widetilde{U_c}(k,t)$.

(A) Find the finite Fourier sine transform of ${\partial u}/{\partial x}$ and express it in terms of $\widetilde{U_c}$.

(B) Find the finite Fourier sine transform of ${\partial^2 u}/{\partial x^2}$ and express it in terms of $\widetilde{U_s}$.

(C) Use your above answers and the principles of the finite Fourier sine transform to solve the following heat conduction equation by integrating both sides of the PDE with respect to x:

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, 0 < x < 2, t > 0 \\ u(0,t) = u(2,t) = 0 \end{cases}$$

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