Please show all intermediate steps and reasoning.

1. (a) (6pts) Find $\lim_{x\to 0} \dfrac{\int_0^{x^2} \sin(t^2)\,dt}{\int_x^0 \sin^{-1}(t^5)\,dt}$

   (b) (7pts) For $1 < a < b$, find $\lim_{t\to\infty}\{\int_0^1 [bx + a(1-x)]^t\,dx\}^{\frac{1}{t}}$

2. Suppose that $f(x)$ is a differentiable function defined on $\mathbf{R}$ with $|f'(x)| < k$, where k is a constant and $0 < k < 1$.

   (a) (4pts) Find $\lim_{x\to\infty} f(x) - x$ and $\lim_{x\to-\infty} f(x) - x$.

   (b) (4pts) Show that $f(x)$ has exactly one fixed point which means that there is exactly one $c \in \mathbf{R}$ such that $f(c) = c$.

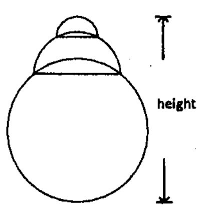
   (c) (7pts) Define $f_2(x) = f(f(x))$ and $f_{n+1}(x) = f(f_n(x))$ for $n \geq 2$. Show that $\{f_n(x)\}$ converges pointwisely to a function $g(x)$. Find $g(x)$.

3. (a) (8pts) Suppose that $f''(x) < 0$ for $x \in I$, where I is an open interval. Given any $a, b \in I$, show that the line segment joining $(a, f(a))$ and $(b, f(b))$ lies under the graph of $y = f(x)$. Also show that the tangent line at $x = a$ lies above the graph of $y = f(x)$ on I.

   (b) (4pts) Let $R_n = \sum_{i=1}^{n} \frac{n}{4n^2+i^2}$, $M_n = \sum_{i=1}^{n} \frac{n}{4n^2+(i-\frac{1}{2})^2}$, and $T_n = \frac{1}{8n} + \sum_{i=1}^{n-1} \frac{n}{4n^2+i^2} + \frac{1}{10n}$. Recognize R_n, M_n, and T_n as approximations of a definite integral, I, and compute $\lim_{n\to\infty} R_n$, $\lim_{n\to\infty} M_n$, and $\lim_{n\to\infty} T_n$.

   (c) (6pts) List R_n, M_n, T_n, and I in increasing order for any $n \in \mathbf{N}$.

4. (12 pts) A hemispherical bubble of radius r_1, $0 < r_1 < 1$, is placed on a spherical bubble of radius 1. Then on the hemispherical bubble a smaller hemispherical bubble of radius r_2, $0 < r_2 < r_1$ is added. Find the maximum height of this tower of three bubbles.



5. (10pts) Find constants r and p such that $\sum_{n=2}^{\infty} \frac{n^p}{\ln n} r^n$ converges.

6. (a) (6pts) Prove Taylor's inequality: If there are positive constants M and d such that $|f''(x)| \leq M$ for $|x-a| \leq d$, then $|f(x) - f(a) - f'(a)(x-a)| \leq \frac{M}{2}|x-a|^2$ for $|x-a| \leq d$.

   (b) (6pts) In the theory of special relativity an object moving with velocity $v(m/s)$ has kinetic energy $K = \frac{m_0c^2}{\sqrt{1-v^2/c^2}} - m_0c^2$, where m_0 is the mass of the object when at rest and $c = 3 \times 10^8 (m/s)$ is the speed of light. However, in the classical Newtonian physics the kinetic energy is $K = \dfrac{1}{2}m_0v^2$. Use Taylor's inequality to estimate the difference in these expressions for K when $|v| \leq 10^3 (m/s)$.

7. Let E be the wedge cut from the cylinder $x^2 + 4y^2 = 4$ by the planes $z = 0$ and $z = x$ with $x \geq 0$.

   (a) (8pts) Evaluate $\iiint_E z\,dV$

   (b) (12pts) Find the area of the boundary surface of E.

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