1. Given an $n\times n$ matrix $\mathbf{A}$ with real entries such that $\mathbf{A}^2=-\mathbf{I}$, where $\mathbf{I}$ is an identity matrix.
   (a) (5%). Show that $\mathbf{A}$ is nonsingular.
   (b) (5%). Show that n is even.
   (c) (5%). Show that $\mathbf{A}$ has no real eigenvalues.

2. Suppose $u=x_1^3-x_2^3+x_3^2,\ v=x_1+x_2+x_3$.
   (a) (5%). Find $\nabla u\times\nabla v$.
   (b) (10%). Suppose $\mathbf{F}=\begin{pmatrix}0\\ f(x_1,x_2,x_3)\\ -2x_1x_3+x_1^3\end{pmatrix}$. Find a $f(x_1,x_2,x_3)$ such that $\nabla\times\mathbf{F}=\text{curl }\mathbf{F}=\nabla u\times\nabla v$.
   (c) (5%). Evaluate the surface integral $\iint_S(\nabla u\times\nabla v)\cdot\mathbf{n}\,dS$, where S is the hemisphere $x_1^2+x_2^2+x_3^2=1,\quad x_3\ge 0$ and $\mathbf{n}$ is the unit outward normal with a non-negative x_3-component.

3. Let $y=y(x)$ be a function of the variable x.
   (a) (15%) Find the general solution of the following differential equation, in power series form, near $x=0$
   $$y''+xy'-3y=x,\quad x>0$$
   (b) (15%) Find the general solution of the following differential-integral equation
   $$x^2y'-4xy+6\int_0^x y(\xi)\,d\xi=x^2,\quad x>0$$

4. The fluctuation of pressure (p) about ambient during one-dimensional acoustic vibration in a tube with length L can be described by the wave equation:
$$c^2\frac{\partial^2 p}{\partial x^2}-\frac{\partial^2 p}{\partial t^2}=0$$
where c is the speed of sound in the fluid.
   (a) (20%) Solve the acoustic wave equation subject to the conditions below
   $$\frac{\partial p}{\partial x}(0,t)=0,\ p(L,t)=0,\ p(x,0)=f(x),\ \frac{\partial p}{\partial t}(x,0)=0$$
   (b) (5%) The fluid velocity u in the x-direction is related to the pressure through the momentum equation (with inviscid approximation) as
   $$\rho\frac{\partial u}{\partial t}=-\frac{\partial p}{\partial x}$$
   where ρ is the fluid density. Find the velocity field u.
   (c) (10%) Plot the first two modes of the velocity (u) distribution in the tube $0\le x\le L$. Where are the points of the maximum velocity?

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