國立臺灣大學 106 學年度碩士班招生考試試題

1. (10 points) State
   (a) Basu's Theorem
   (b) Rao-Blackwell Theorem

2. (10 points) Let X have a uniform distribution $U(0,1)$, and let the conditional distribution of Y, given that $X = x$, be $U(0, e^x)$.
   (a) Calculate the marginal pdf of Y.
   (b) Compute $E(Y|x)$, the conditional mean of Y, given that $X = x$.

3. (10 points) Let $X_1, X_2, \ldots, X_n$ be *iid* $f(x|\theta)$ with $E(X_i), E(X_i^2) < \infty$. Let $\hat{\theta}_n$ be an estimator of θ.
   (a) What is the definition of an unbiased estimator of θ?
   (b) Show that
   $$E\left[(\hat{\theta}_n - \theta)^2\right] = \left[E(\hat{\theta}_n) - \theta\right]^2 + E\left[\left(\hat{\theta}_n - E(\hat{\theta}_n)\right)^2\right] = c_n + d_n$$
   (c) What conditions on c_n and d_n imply that $\hat{\theta}_n$ is a consistent estimator of θ? Prove your answer.

4. (10 points) Let $X_1, \ldots, X_n$ be a random sample from a population with pdf
   $$f(x|\theta) = \theta x^{\theta-1}, \; 0 \leq x \leq 1, \; 0 < \theta < \infty.$$
   (a) Please find the MLE of θ, and show that its variance converges to 0 as $n \to \infty$.
   (b) Please find the method of moment estimator of θ.

5. (a) (5 points) Let $X_1, X_2, \ldots, X_n$ be *iid* Poisson distribution with mean λ. Please find the UMVUE (*uniform minimum variance unbiased estimator*) of $(\lambda - 1)(\lambda - 2)$.
   (b) (5 points) Let $X_1, X_2, \ldots, X_n$ be *iid* Uniform distribution $(0, \theta)$. Please find the UMVUE of θ^{-2}.

6. (10 points) Let X be one observation from a distribution with p.d.f. given by
   $$f(x) = \frac{e^{(x-\theta)}}{[1 + e^{(x-\theta)}]^2},$$
   where $-\infty < x < \infty$; and $-\infty < \theta < \infty$. Find a uniformly most powerful test for testing $H_0 : \theta \leq 0$ and $H_1 : \theta > 0$.

見背面

7. (10 points) Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with p.d.f. given by

$$f(x) = \theta x^{\theta-1},$$

where $0 < x < \infty$; and $\theta > 0$. Find a uniformly most powerful test for testing $H_0 : \theta = 6$ against $H_1 : \theta < 6$.

8. (30 points) Let $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_m$ be independent random samples from the normal distributions $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$, respectively.

   (a) (10 points) If μ_1 and μ_2 are unknown, show that the likelihood ratio test for testing $H_0 : \sigma_1^2 = \sigma_2^2$ against $H_1 : \sigma_1^2 \neq \sigma_2^2$ can be based on the following random variable:

$$F = \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2/(n-1)}{\sum_{k=1}^{m}(Y_k - \bar{Y})^2/(m-1)},$$

   where

$$\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n} \text{ and } \bar{Y} = \frac{\sum_{k=1}^{m} Y_k}{m}.$$

   (b) (10 points) If $\sigma_1^2 = \sigma_2^2$, show that the random variable F in (a) is independent of the random variable T given by

$$T = \frac{\sqrt{\frac{nm}{n+m}}(\bar{X} - \bar{Y})}{\sqrt{\frac{\sum_{i=1}^{n}(X_i-\bar{X})^2 + \sum_{k=1}^{m}(Y_k-\bar{Y})^2}{n+m-2}}}.$$

   (c) (10 points) Show that the likelihood ratio test for testing $H_0 : \mu_1 = \mu_2$ and $\sigma_1^2 = \sigma_2^2$ against all alternatives can be based on the following random variable:

$$W = \frac{\left[\sum_{i=1}^{n}(X_i - \bar{X})^2/n\right]^{\frac{n}{2}} \left[\sum_{k=1}^{m}(Y_k - \bar{Y})^2/m\right]^{\frac{m}{2}}}{\left\{\left[\sum_{i=1}^{n}(X_i - \hat{\mu})^2 + \sum_{k=1}^{m}(Y_k - \hat{\mu})^2\right] \Big/ (n+m)\right\}^{\frac{n+m}{2}}},$$

   where $\hat{\mu} = (n\bar{X} + m\bar{Y})/(n+m)$.

試題隨卷繳回