

※ 注意：請於試卷內之「非選擇題作答區」作答，並應註明作答之題號。

From Problem 1 to Problem 4, your answer will be considered correct only if all the true statements are selected.

1. (5%) Let $A, B \in M_{n \times n}(F)$, where $F = \mathbb{C}$ or \mathbb{R} . Which of the following statements are true?

- (a) $\text{tr}(AB) = \text{tr}(BA)$
- (b) $\text{tr}(AB) = \text{tr}(A)\text{tr}(B)$
- (c) $\text{tr}(B^{-1}AB) = \text{tr}(A)$
- (d) $\text{tr}(A^k) = (\text{tr}(A))^k$
- (e) $\text{tr}(A \pm B) = \text{tr}(A) \pm \text{tr}(B)$

2. (5%) Let $A, B \in M_{n \times n}(F)$, where $F = \mathbb{C}$ or \mathbb{R} . Which of the following statements are true?

- (a) $\det(AB) = \det(BA)$
- (b) $\det(AB) = \det(A)\det(B)$
- (c) $\det(B^{-1}AB) = \det(A)$
- (d) $\det(A^k) = (\det(A))^k$
- (e) $\det(A \pm B) = \det(A) \pm \det(B)$

3. (5%) Which of the following statements are true?

- (a) If A and B are invertible matrices, then the matrix $C = A^{-1} + B^{-1}$ is also invertible.
- (b) If R is a rectangular matrix, then $A = R^T R$ is positive definite.
- (c) Every orthogonal set is linearly independent.
- (d) $T = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix}$, where T_{11} is a $p \times p$ matrix, T_{22} is a $q \times q$ matrix, and T_{12} is a $p \times q$ matrix. The set of eigenvalues of T is the union of the sets of eigenvalues of T_{11} and T_{22} .
- (e) If $T: V \rightarrow W$ is a linear transformation with V and W are vector spaces over a field F . Then, $\ker(T)$ is a vector space.

4. (10%) Let vectors $x \in \mathbb{R}^n$ be represented as $x = (x_1, x_2, \dots, x_n)^T$. Which of the following sets is a subspace of \mathbb{R}^n ?

- (a) $\{x \in \mathbb{R}^n \mid \sum_{i=1}^n x_i^2 = n^2\}$
- (b) $\{x \in \mathbb{R}^n \mid x_1 = 1\}$
- (c) $\{x \in \mathbb{R}^n \mid \sum_{i=1}^n \alpha_i x_i = \min_i(\alpha_i)\}$ for some known constants $\alpha_i \in \mathbb{R}$.
- (d) $\{x \in \mathbb{R}^n \mid x_{2i} = x_{2i-1}, \forall i = 1, \dots, n/2\}$, where n is assumed to be even in this case.
- (e) $\{x \in \mathbb{R}^n \mid x_{2i} - x_{2i-1} = 1, \forall i = 1, \dots, n/2\}$, where n is assumed to be even in this case.

5. (5%) Let N be an $n \times n$ matrix over \mathbb{R} or \mathbb{C} such that $N^k = 0$ for some integer k . Is $I+N$ invertible? If yes, find its inverse. If not, provide the reason why.

6. (5%) Define an inner product in \mathbb{R}^2 as $\langle u, v \rangle = \frac{1}{4}u_1v_1 + \frac{1}{9}u_2v_2$. What is the equation of the unit circle?

7. (5%) Please find the eigenvectors of $(I + A)^{100}$ given $A = \begin{bmatrix} -4 & -5 \\ 10 & 11 \end{bmatrix}$.

見背面

8. (10%) Please find the determinant of A .

$$A = \begin{pmatrix} 1+x_1 & x_2 & x_3 & \cdots & x_n \\ x_1 & 1+x_2 & x_3 & \cdots & x_n \\ \vdots & \vdots & 1+x_3 & \ddots & \vdots \\ x_1 & x_2 & x_3 & \cdots & 1+x_n \end{pmatrix}$$

9. (10%) Consider

$$x_1 + x_2 + \cdots + x_n < r,$$

where $x_i \geq 0$ for $1 \leq i \leq n$. When $n=4$ and $r=8$, the number of nonnegative integer solutions is _____.

10. (5%) The cycle decomposition of the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 6 & 5 & 8 & 3 & 1 & 7 & 4 \end{pmatrix}$$

is _____.

11. (10%) The solution to the recurrence equation

$$a_n = 2a_{n-1} + n$$

with $a_0 = 4$ is $a_n =$ _____.

12. (10%) The generating function in partial fraction decomposition for the above recurrence equation is _____ (Note that expressions like

$$\frac{x-8}{(x-3)^2} - \frac{9}{x-1}$$

are *not* partial fraction decompositions.)

13. (5%) Let $p(m)$ denote the number of partitions of $m \in \mathbb{Z}^+$. For example, the number of partitions of $m=5$ is $p(5) = 7$:

$$5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1, 1+1+1+1+1$$

The generating function for $\{p(n)\}_{n=0,1,\infty}$ is _____.

14. (5%) A Boolean function f is **self-dual** if

$$f(x_1, x_2, \dots, x_m) = f(\neg x_1, \neg x_2, \dots, \neg x_m).$$

There are _____ self-dual Boolean functions of m variables.

15. (5%) For any binary tree with n nodes and i internal nodes, the relation between n and i is _____ $\leq i$.

試題隨卷繳回