Notation: $\mathbb{R}$ is the set of real numbers and $\mathbb{C}$ is the set of complex numbers.

Problem 1 (15 pts). Let $<,>$ be the standard inner product on $\mathbb{R}^3$ given by $<v,w>= a_1a_2 + b_1b_2 + c_1c_2$ if $v = (a_1, b_1, c_1)$ and $w = (a_2, b_2, c_2)$. Let W be the subspace in $\mathbb{R}^3$ given by

$$W = \{(x, y, z) \in \mathbb{R} \mid 2x + 7y = 0,\ x - 2y + z = 0\}.$$

Find an orthonormal basis of W. Namely, find a basis $\{w_1, w_2\}$ of W such that $<w_1, w_1>=<w_2, w_2>= 1$ and $<w_1, w_2>= 0$.

Problem 2 (20 pts). Let

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 2 & -1 \\ -1 & 1 & 4 \end{pmatrix}.$$

(1) Compute the characteristic polynomial of A.
(2) Find an invertible $P \in M_3(\mathbb{R})$ such that $P^{-1}AP$ is diagonal.

Problem 3 (25pts). Let V be a finite dimensional vector space over $\mathbb{R}$ and let $A : V \to V$ be a $\mathbb{R}$-linear transformation. Prove that

(1) (10 pts) if $A^k = 0$ for some positive integer k, then $I - A$ is invertible, where I is the identity map.
(2) (15 pts) V is generated by kernel of A^k and the image of A^k for some k. In other words, prove $V = \operatorname{Ker} A^k + \operatorname{Im} A^k$ for some k.

Problem 4 (20pts). Let $\mathrm{L} : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ be the linear transformation defined by

$$\mathrm{L}(X) = \begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix} X - X \begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix}$$

(1) Find the dimension of the kernel of L.
(2) Find a basis for the image of L.

Problem 5 (20 pts). If $A \in M_n(\mathbb{C})$ such that $AA^* = A^*A$ and $v \in \mathbb{C}^n$ is a column vector, prove that

(1) $A^2v = 0$, then $Av = 0$.
(2) If $A^kv = 0$ for some $k \geq 1$, then $Av = 0$.
(3) Show that the minimal polynomial of A has distinct roots.