國立臺灣大學101學年度碩士班招生考試試題

1. (a) Let $A=\begin{bmatrix}\alpha & -4\\ 3 & \beta\end{bmatrix}$, suppose $u_1=\begin{bmatrix}4\\3\end{bmatrix}$, and $u_2=\begin{bmatrix}1\\1\end{bmatrix}$ are eigenvectors of A.

   (i) (5%) Find α and β. Compute $tr(A^5)$.

   (ii) (7%) Assume B is another 2×2 matrix, whose eigenvectors are the same with A, i.e., u_1 and u_2 with corresponding eigenvalues $\lambda_1=1$ and $\lambda_2=-1$. Calculate B^{10}.

   (b) Assume that C is a real nonsingular symmetric $n\times n$ matrix with distinct eigenvalues.

   (i) (6%) Show that eigenvalues of C are all real and eigenvectors are all mutually orthogonal.

   (ii) (6%) Solve $Tx=y$ where $T=C^2$ and x,y both are $n\times 1$ column vectors.

   (c) Assume that matrix A is idempotent (i.e., $A^2=A$).

   (i) (3%) Find all eigenvalues of A.

   (ii) (3%) Show that $I-A$ is also idempotent.

   (ii) (3%) Show that $(I-2A)^{-1}=I-2A$.

2. Consider the equation describing the small angular displacement θ, of a pendulum in a viscous liquid. A force balance gives

$$\frac{d^2\theta}{dt^2}+\omega^2\theta+\sigma\frac{d\theta}{dt}=f(t)$$

   where $\omega>0$. The term $\sigma\frac{d\theta}{dt}$ is damping due to friction within the liquid.

   (a) (4%) Make the equation dimensionless defining $\tau=\omega t$ to show that the solution $\theta(\tau)$ depends on a single dimensionless parameter $\beta=\frac{\sigma}{\omega}$.

   (b) (10%) Find the solution $\theta(\tau)$ and $\theta'(\tau)$ for $\beta=1$ and $\beta=3$ for the homogeneous equation when there is no forcing, i.e. $f(t)=0$, with the boundary conditions $\theta(0)=\theta_0,\ \theta'(0)=0$.

   (c) (7%) Find the general solution $\theta(\tau)$ for $\beta=1$ when there is a decaying forcing $f(t)=e^{-0.5\omega t}$, where ω is real constant.

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3. Consider the differential eigenvalue problem

$$x^2\phi'' + 2x\phi' + \lambda x^2\phi = 0$$, with the boundary condition

$$\phi'(0) = 0,\ \phi(1) = 0,$$

where λ is the eigenvalue.

(a) (4%) Is this problem of Sturm-Liouville type? If so, put the problem in "standard form".

(b) (9%) Find the eigenvalues and eigenfunctions for this problem (Hint: Conside $\phi = \dfrac{f}{x}$ and solve for f).

4. Consider a partial differential equation of $u(x, y)$

$$3\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} + 3u = 4\sin(x - 2y)$$

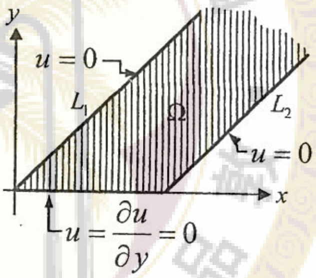
in a semi-infinite strip Ω, as shown in the figure, bounded by lines L_1, L_2 and x-axis:

$L_1: x - 2y = 0$, $L_2: x - 2y = \pi$, x-axis: $y = 0$

The boundary conditions and initial conditions are

$u = 0$ along L_1 & L_2,

and $u = \dfrac{\partial u}{\partial y} = 0$ along x-axis.



(a) (16%) Show that by using change of variables

$\xi = x - 2y$, $\eta = y$

will reduce the partial differential equation into the form of

$$\frac{\partial^2 u}{\partial \xi^2} - \frac{\partial^2 u}{\partial \eta^2} + \alpha u = F(\xi, \eta)$$

Find also the constant coefficient α and the function $F(\xi, \eta)$.

(b) (17%) Find the solution of $u(x, y)$.