國立臺灣大學101學年度碩士班招生考試試題

1. (25 points) Consider the fixed point iteration

$$x_{n+1} = (\alpha+1)x_n - x_n^2, \qquad n = 0, 1, \ldots,$$

where α satisfying $1/2 \le \alpha \le 1$ is given.

(a) Show that the iteration converges for any initial guess x_0 satisfying $\alpha - 1/5 \le x_0 \le \alpha + 1/5$.

(b) Assume that the iteration converges, find the value of α for which the method converges quadratically.

2. (25 points)

(a) Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$. Compute the LU factorization of A without pivoting.

(b) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and assume $a \neq 0$. Let $A = LU$ be the LU factorization of A without pivoting. Show that the pivots of A are positive if and only if A is symmetric positive definite.

3. (25 points) Consider the quadrature of the form

$$\int_{-1}^{1} |x| f(x)\, dx \approx A\left(f(x_1) + f(x_2)\right).$$

Determine the constants A, x_1, and x_2 so that this quadrature has the highest precision with respect to the function $f(x)$. What is the degree of precision ?

4. (25 points) Let $a = x_0 < x_1 < \cdots < x_m = b$ be a mesh on $[a, b]$. Let l_i be defined on $[a, b]$ as follows:

1. l_i is piecewise linear on $[a, b]$,
2. $l_i(x_i) = 1$,
3. $l_i(x_j) = 0$ $(j \neq i)$.

(a) Let $f(x)$ be continuous on $[a, b]$. Determine the values of coefficients c_i such that the function

$$l(x) = \sum_{i=0}^{m} c_i l_i(x)$$

satisfies $l(x_i) = f(x_i)$.

(b) Let $h = \max_i\{x_{i+1} - x_i\}$. Let f be twice differentiable on $[a, b]$. Derive an upper bound on $|f(x) - l(x)|$, and show that $\lim_{h\to 0} |f(x) - l(x)| = 0$. Here l is the function defined in part a.

試題隨卷繳回