# 國立臺灣大學101學年度碩士班招生考試試題



You have to show the details of your proofs and calculations.

1. (20 points) Suppose $\{f_n(x)\}_{n=1}^{\infty}$ is a sequence of continuous functions defined on a compact set $A \subset \mathbb{R}^n$ such that
   (1) $|f_n(x)| < M$ for $x \in A$, $n = 1, 2, 3, \cdots$.
   (2) $\{f_n\}_{n=1}^{\infty}$ is equi-continuous. i.e., for any $\epsilon > 0$, there is a $\delta > 0$ such that if $x, y \in A$ and $|x - y| < \delta$, then $|f_n(x) - f_n(y)| < \epsilon$ for all $n = 1, 2, 3, \cdots$.

   Prove that $\{f_n\}_{n=1}^{\infty}$ has a uniformly convergent subsequence.

2. (30 points)
   (1) Calculate
   $$\int\int_{\mathbb{R}^2} e^{-(4x^2+4xy+5y^2)} dxdy$$
   (2) Calculate
   $$\lim_{n\to\infty} \int_0^1 \frac{1-e^{-nx}}{\sqrt{x}} dx.$$
   (3) Calculate
   $$\int\int\int_D z\sqrt{x^2+y^2}\, dxdydz,$$
   where
   $$D = \left\{(x,y,z)\middle| 1 \le x^2 + y^2 \le 2,\ 1 \le z \le 2\right\}.$$

3. (15 points) Let $A \subset R^2$ be open and $f(x_1, x_2) : A \to \mathbb{R}$. Prove that if both $\frac{\partial f}{\partial x_1}$ and $\frac{\partial f}{\partial x_2}$ exist and are continuous at $(y_1, y_2) \in A$, then $f$ is differentiable at $(y_1, y_2)$.

4. (20 points) Let $f(x) : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous vector-valued function. Suppose that there exists a positive constant $\lambda < 1$ such that $|f(x) - f(y)| \le \lambda |x - y|$, then there exists a unique point $x_0 \in R^n$ such that $f(x_0) = x_0$.

5. (15 points) Let $f(x, y)$ be defined as
   $$f(x,y) = \begin{cases} \dfrac{x^2y^2}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{at } (x,y) = (0,0). \end{cases}$$
   Is $f(x, y)$ differentiable at $(0, 0)$ ? Prove or disprove your answer.

試題隨卷繳回