1. (30%) Suppose a random value y follows the g distribution, i.e., y can be expressed as a function of a normally distributed variable as follows.

$$y = \delta + \lambda \frac{\exp(gz) - 1}{g},$$

where δ and g are real numbers, λ is a positive real number, and z follows the normal distribution with mean to be μ and standard deviation to be σ. Given the constraint $[g(y-\delta)/\lambda + 1] > 0$, solve the following problems.

(a) (4%) Express z in terms of y, i.e., find the function $h(y) = z$.

(b) (4%) With the function $h(y)$, derive the density function of y, $f(y)$, according to the definition $f(y) = \frac{1}{\sigma\sqrt{2\pi}} |h'(y)| e^{-\frac{1}{2\sigma^2}(h(y)-\mu)^2}$, where μ and σ are the mean and the standard deviation of z.

(c) (8%) Derive the expectation of y, $E(y)$.

(d) (8%) Derive the variance of y, $\text{var}(y)$.

(e) (6%) Show that when g is zero, y follows a normal distribution. [Hint: Consider the result of $\lim_{g \to 0} \frac{\exp(gz) - 1}{g}$.]

2. (8%) Find the maximum and minimum values of $f(x, y, z) = x + 3y - z$ subject to $4x^2 + 2y^2 + z^2 = 4$.

3. (12%) A 10,000-cubic-foot-room has 500 smoke particles per cubic foot. A ventilation system is turned on that each minute brings in 500 cubic feet of smoke-free air, while an equal volume of air leaves the room. Also, during each minute, smokers in the room add a total of 10,000 particles of smoke to the room. Assume that the air in the room mixes thoroughly.

(a) (4%) Find a differential equation and initial condition that govern the total number $y(t)$ of smoke particles in the room after t minutes.

(b) (4%) Solve this differential equation.

(c) (4%) Find how soon the smoke level will fall to 100 smoke particles per cubic foot. (If the answers are with decimal numbers, please round to the nearest hundredth.)

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4. Evaluate:

(a) (5%) $\phi(\alpha)=\int_0^\infty \alpha e^{-\alpha x}\,dx$ for $\alpha>0$.

(b) (10%) Evaluate $\int_0^1 e^{x^2}\,dx$, approximately by using Taylor's theorem of the mean and estimate the maximum error.

5. Let φ be a continuous real function on $[0,\alpha]$. Assume that $\varphi(x)>0$ if $0<x\le\alpha$ and that $\varphi(x)\sim Ax^r$ as $x\to 0$ $(A>0, r\ge 0)$. For all $t>0$ set

$$F(t)=\int_0^\alpha \frac{dx}{t+\varphi(x)}.$$

(a) (10%) Show that if $r>1$, then as $t\to 0$

$$F(t)\sim \frac{\pi}{rA^{1/r}\sin(\pi/r)}\cdot\frac{1}{t^{1-1/r}}.$$

(b) (10%) Show that if $r=1$, then as $t\to 0$

$$F(t)\sim \frac{1}{A}\log\frac{1}{t}.$$

6. Suppose that $a_n\to c$ as $n\to\infty$ and that $\{\alpha_i\}_{i=1}^\infty$ is a sequence of positive terms for which $\sum_{i=1}^n \alpha_i\to\infty$ as $n\to\infty$.

(a) (10%) Show that

$$\frac{\sum_{i=1}^n \alpha_i a_i}{\sum_{i=1}^n \alpha_i}\to c$$ as $n\to\infty$. In particular, if $\alpha_i=1$ for all i, then

$$\frac{1}{n}\sum_{i=1}^n a_i\to c \quad \text{as} \quad n\to\infty.$$

(b) (5%) Show that the converse of the special case in (a) does not always hold by giving a counterexample of a sequence $\{a_n\}_{n=1}^\infty$ that does not converge, yet $\frac{1}{n}\sum_{i=1}^n a_i$ converges as $n\to\infty$.

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