1. Suppose

$$\mathbf{A}=\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix},\qquad \mathbf{B}=\begin{pmatrix}1 & 1 & 1\\ 1 & e^{\frac{i2\pi}{3}} & e^{\frac{i4\pi}{3}}\\ 1 & e^{\frac{i4\pi}{3}} & e^{\frac{i8\pi}{3}}\end{pmatrix},$$

where $i^2=-1$.

(a). (5%). Find $\mathbf{A}^{-1}$.

(b). (6%). Find $\mathbf{B}^{-1}$.

2. Suppose $\mathbf{A}$ is a real square matrix. $\mathbf{A}$ is positive definite if $\mathbf{u}^T\mathbf{A}\mathbf{u}>0$ for all real vectors $\mathbf{u}\neq\mathbf{0}$. Consider a 2×2 real symmetric matrix $\mathbf{A}$ with

$$\mathbf{A}\mathbf{x}^{(1)}=\lambda_1\mathbf{x}^{(1)},\qquad \mathbf{A}\mathbf{x}^{(2)}=\lambda_2\mathbf{x}^{(2)},$$

where λ_1 and λ_2 are eigenvalues of $\mathbf{A}$ and $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are linearly independently vectors.

Assume $\mathbf{A}$ is positive definite.

(a) (5%) Show that $\lambda_1>0$, $\lambda_2>0$.

(b) (10%). Suppose $\lambda_1=1$, $\lambda_2=4$, and

$$\mathbf{x}^{(1)}=\begin{pmatrix}1\\1\end{pmatrix},\qquad \mathbf{x}^{(2)}=\begin{pmatrix}1\\-1\end{pmatrix}.$$

Find a positive definite, symmetric matrix $\mathbf{B}$ such that $\mathbf{B}^2=\mathbf{A}$.

(c) (8%). Suppose there exist two positive definite, symmetric matrices $\mathbf{U}$ and $\mathbf{V}$ such that $\mathbf{U}^2=\mathbf{A}$ and $\mathbf{V}^2=\mathbf{A}$. Show that $\mathbf{U}$=$\mathbf{V}$.

3. Solve the following ordinary differential equations:

(a) (8%) $y'=\sqrt{|y|},\ y(0)=0.$

(b) (8%) $x^2y''-3xy'+4y=0,\ y(1)=1,\ y(2)=2.$

(c) (8%) $y''+y=\sec x,\ y(0)=1,\ y(\pi/4)=-1.$

(d) (8%) Consider the differential equation: $x(x-1)y''+(3x-1)y'+y=0.$
Verify that the 1st homogeneous solution is $y_1(x)=1/(1-x)$. Find the 2nd homogeneous solution $y_2(x)$.

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4. Consider the partial differential equation:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=1, \quad 0<x<1, \quad 0<y<1;$$

$$u(x,0)=2, \quad u(x,1)=2;$$

$$u(0,y)=2, \quad u(0,1)=3.$$

(a) (15%) Solve for $u(x,y)$.

(b) (3%) Interpret the physical meaning of the above problem.

(c) (2%) What is the value u at $x=0.5$ and $y=0.5$?

5. (8%) For a scalar function $u(x,y,z)$ and a vector function $\mathbf{V}(x,y,z)$, express ∇u, $\nabla^2 u$, $\nabla\cdot\mathbf{V}$, $\nabla\times\mathbf{V}$ and $\nabla^2\mathbf{V}$ in detailed forms involving derivates in Cartesian coordinates. Also express ∇u in cylindrical coordinates.

6. (6%) Evaluate $\oint_c (x^2-y^2)dx+(x^2+y^2)dy$ if C is the square with vertices (0, 0), (1, 0), (1,1) and (0,1).