1. Find the solutions: (20%)

(1) $y'' + y = F(x)$, $0 < x < 1$; $y(0) = A$, $y'(1) = B$, A、B are constants

(2) $y'' - 4y = e^{2x} + \sin 3x$

2. The following is the Fibonacci series, (20%)

1 , 1 , 2 , 3 , 5 , 8 , 13 , 21 , 34 , 55 , 89 , ⋯

$a_0 = 1$, $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, $a_4 = 5$, $a_5 = 8$, ...

(a) Please find all mathematical rules.

(b) Please find $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = ?$

(c) Please prove $a_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]$

(d) If $F(x) = \frac{1}{1-x-x^2} = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$

Please find a_0, a_1, a_2, …, a_n, …

3. Show that the give vector functions are linearly independent solutions of the given differential equation; obtain the general solution and the particular solution satisfying the given initial condition: (20%)

$$\frac{d\mathbf{X}}{dt} = \begin{bmatrix} -2 & 2 \\ -15 & 9 \end{bmatrix}\mathbf{X},\ \mathbf{X}_1 = \begin{bmatrix} 2e^{3t} \\ 5e^{3t} \end{bmatrix},\ \mathbf{X}_2 = \begin{bmatrix} e^{4t} \\ 3e^{4t} \end{bmatrix},\ \mathbf{X}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

4. If we consider scalar field of pollutant concentration in space given by a scalar function $f(P) = f(x, y, z)$, what is the gradient of this scalar field function? What is the divergence of this gradient? Please explain physical meanings of the gradient and divergence. (20%)

5. Consider the following model in which $u = u(x,t)$: $u_t = k u_{xx}$, $0 < x < 1$, $0 < t$
$u_x(0,t) = 0$ & $u(1,t) = u(0,t)$, $0 < t$; $u(x,0) = 1414 \sin \pi x$, $0 < x < 1$

The appropriate eigenvalue problem is (if $u(x,t) = X(x) \cdot T(t)$)

$X'' + \lambda^2 X = 0$, $X'(0) = 0$, $X(1) = X(0)$

Find: the eigenvalue λ_n and eigenfunctions $X_n(x)$. (20%)