I. (20%) Let $\mathbf{x}(\mathbf{u},\mathbf{v})$ be a canonical parametrization of a smooth surface of revolution in $\mathbf{R}^3$ defined by

$$\mathbf{x}(u,v) = (g(u), h(u)\cos v, h(u)\sin v).$$

Here u is in an interval I with $h(u) > 0$ and $(g')^2 + (h')^2 = 1$.

(i) Show that the Gaussian curvature K is

$$\mathrm{K} = -\frac{h''}{h}.$$

(ii) Find the functions $g(u)$ and $h(u)$ when $\mathrm{K}(u)$ is a positive constant $\frac{1}{c^2}$.

II. (40%) Consider the smooth Riemannian 2-manifold $D = \{(x,y) \in \mathbf{R}^2; x^2 + y^2 < 1\}$ with a metric

$$\frac{dx^2 + dy^2}{g^2(x,y)}.$$

Here

$$g(x,y) = \frac{1}{2}[1 - (x^2 + y^2)].$$

(i) Find the frame field $\{E_1, E_2\}$ and dual coframe field $\{\theta_1, \theta_2\}$.

(ii) Show that the connection 1-form $w_{12} = g_y\theta_1 - g_x\theta_2$.

(iii) Show that the Gaussian curvature

$$\mathrm{K} = -1.$$

(iv) Let Δ be a geodesic triangle in D. Show that

$$area(\Delta) = \pi - (i_1 + i_2 + i_3).$$

Here i_1, i_2, i_3 are the interior angles at the vertices of Δ.

III. (20%) Let φ be a positive smooth function defined on all of $\mathbf{R}$. Let M be the smooth Riemannian 1-manifold with inner product on the tangent space $T_x(M) \cong \mathbf{R}$ given as

$$u \cdot v = \varphi(x)uv.$$

(i) Show that M is isometric to an open interval in $\mathbf{R}$ with its usual differentiable structure.

(ii) Find a necessary and sufficient condition on φ in order that M is isometric to $\mathbf{R}$ with its usual differentiable structure.

IV. (20%) Let Σ be a closed orientable smooth surface in $\mathbf{R}^3$.

(i) For any two principal curvatures κ_1, κ_2 with the Gaussian curvature $\mathrm{K} = \kappa_1\kappa_2$ and the mean curvature $\mathrm{H} = \frac{1}{2}(\kappa_1 + \kappa_2)$, show that

$$\mathrm{H}^2 - \mathrm{K} \geq 0.$$

(ii) Show that

$$\int_\Sigma \mathrm{H}^2 dS \geq 4\pi.$$