1. (25 points) Find the general solution of the equation

$$y'' - \frac{2+x+\alpha x}{x(1+x)}y' + \frac{2+x+\alpha x}{x^2(1+x)}y = 0, x > 0,$$

where α is a real number. (Hint: $y = x$ is a solution. Use the method of reduction of order.)

2. Assume that $y(t)$, $y'(t)$, and $y''(t)$ are continuous on $[0, \infty)$, $y(0) = 0$, and $y(t) \geq 0$.
(a) (9 points) Show that $y(t) = 0$ on $[0, \infty)$ if $y' \leq y$.
(b) (8 points) Show that $y(t) = 0$ on $[0, \infty)$ if $y' \leq y^2$.
(c) (8 points) Is it true that $y(t) = 0$ on $[0, \infty)$ if $y'' \leq y$ and $y'(0) = 0$?

3. (25 points) Find the general solution of the linear system

$$\mathbf{x}'(t) = \begin{pmatrix} 1 & 4 \\ -1 & 1 \end{pmatrix} \mathbf{x}(t) + \begin{pmatrix} e^t \cos(2t) \\ 0 \end{pmatrix}.$$

4. Suppose $y(t)$ and $z(t)$ satisfy the equations

$$y' = y(1 - \frac{y}{5}), y(0) = 2; z' = z(1 - \frac{z}{y}), z(0) = 2.$$

Show that
(a) (15 points) $0 < y(t) < 5$, $\lim_{t\to\infty} y(t) = 5$, and $\lim_{t\to-\infty} y(t) = 0$.
(b) (10 points) $z(t) < y(t)$ for $t > 0$.