1. Basic univariate calculus (9 points each)

(a) $y = e^{\sin(x^2)}$, find dy/dx

(b) Find the Taylor series approximation of $(1-x)^{-1}$ about $x_0 = 0$

(c) The Coriolis parameter f is defined as $f = 2\Omega \sin\theta$, where $\Omega = |\vec{\Omega}| = 7.29 \times 10^{-5}$ sec^{-1} is the Earth's rotation rate and θ is latitude. Find the Taylor series expansion of the Coriolis parameter f about a latitude θ_0.

(d) Find $\lim_{x\to 0} \frac{\cos(x)-1}{x^2}$

(e) Find the integral $\int_{x_1}^{x_2} x^2 \cos(x)\, dx$

(f) The sea surface height h of nonlinear mesoscale eddies in the ocean are often characterized by a Gaussian function

$$h = Ae^{-r^2/L^2},$$

where the amplitude A is a constant, r is the radial distance from the eddy center, and L is a length scale that characterizes the eddy size. The mesoscale eddies are nearly in geostrophic balance, meaning that the eddy velocity (i.e. rotational speed) is proportional to the height gradient dh/dr.

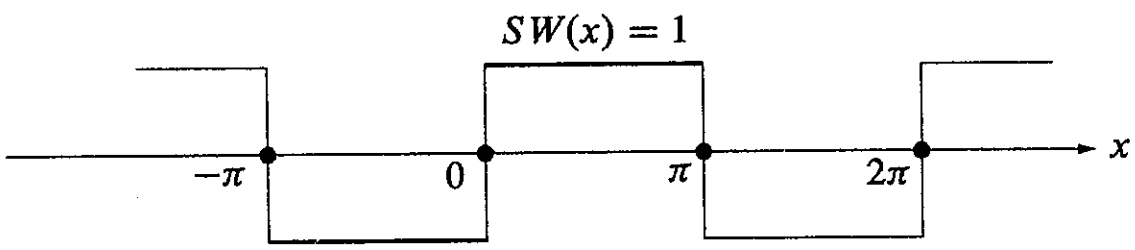
Find the radius at which the rotational speed of the eddy is maximum.

(g) Following (f), find the area integral of the Gaussian function h.

2. (15 points) A periodic square wave is defined as

$$SW(x) = \begin{cases} -1 & if -\pi \le x < 0 \\ 0 & if\ x = 0 \\ 1 & if\ 0 < x \le \pi \end{cases},$$

with $SW(x + 2\pi) = SW(x)$.



見背面

The Fourier sine series is

$$S(x) = b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \cdots = \sum_{n=1}^{\infty} b_n \sin nx.$$

Find the Fourier sine coefficients b_k for the square wave $SW(x)$.

3. (22 points)
The surface temperature of the ocean is maintained by a balance between heating and cooling. The heating occurs both by solar radiation and by downward longwave radiation from the atmosphere. The cooling, on the other hand, increases with the temperature—a hot object cools down faster than a warm one. If for simplicity we suppose that the cooling rate varies linearly with temperature, then we can model the surface temperature by the equation

$$C\frac{dT}{dt} = S - \lambda T$$

Here, S is the heating source, T is the temperature, and t is time. The parameter C represents the heat capacity of the system, and λ is a constant that determines how fast the body cools when it is hot.

(a) Suppose that $T = T_0$ at $t = 0$, and the heating S is a constant. Find the solution for $T(t)$. (8 points)

(b) Now let us consider that the heating is cyclic (e.g. like the annual cycle of solar radiation), with $S = S_0 \cos(\omega t)$, where ω is the frequency of the heating and S_0 is its amplitude. Find the solution for $T(t)$. (8 points)

(c) Following (b), find the approximate solution of $T(t)$ when the heat capacity C is small and/or the forcing frequency ω is very low (i.e. slow heating changes) such that $\lambda \gg C\omega$.

Interpret the meaning of the result you obtain. (6 points)

試題隨卷繳回