Any device with computer algebra system is prohibited during the exam.

PART 1 : Fill in the blanks.

- Please ensure that each answer is clearly labeled with the corresponding blank number.
- Please note that only the final answers will be graded, and each blank is worth 5 points.

1. (a)

$$\lim_{x\to 0} |x - \tan x|^{\frac{1}{\ln|x|}} = \underline{\quad (1) \quad}.$$

(b) The 3rd degree Taylor polynomial of $(1+3x)^e$ at $x=0$ is $\underline{\quad (2) \quad}$.

$$\lim_{x\to\infty} \left(x^2 \left(1 + \frac{3}{x}\right)^e - x^2 - 3ex \right) = \underline{\quad (3) \quad}.$$

2. Assume that the equation

$$2y^x = x + y$$

defines y as an implicit function of x, denoted by $y = f(x)$, near the point $(x,y) = (0,2)$. Then $f'(0) = \underline{\quad (4) \quad}$. The tangent plane of the graph of $g(x,y) = 2y^x - x - y + 1$ at $(0,2,1)$ is $\underline{\quad (5) \quad}$.

3. Compute the integrals.

$$\int \frac{1}{\sqrt{4x^2+4x}}\, dx = \underline{\quad (6) \quad}.$$

$$\int_0^1 \frac{x^2}{(x^2+1)^2}\, dx = \underline{\quad (7) \quad}.$$

$$\iint_D \frac{y}{x}\, dA = \underline{\quad (8) \quad},$$ where D is the region in the first quadrant bounded by $x^2+y^2 = \frac{1}{4}$, $x^2+y^2 = x$, $y = x$, and $y = 0$.

4. Let $\mathbf{F}(x,y,z) = xz\mathbf{i} + yz\mathbf{j} + (-z + ey^2)\mathbf{k}$ and S be the part of the cylinder $x^2+y^2=1$ between planes $z=0$ and $z = 2+x$ with outward orientation. A parametrization of the surface S is $\underline{\quad (9) \quad}$. The flux of $\mathbf{F}$ through S is $\underline{\quad (10) \quad}$.

PART 2 :

- Please solve the following problems and provide computations as well as explanations.
- Partial credits are allocated according to the level of completeness in your work.

1. Suppose that $f(x)$ is a function defined on $\mathbf{R}$ satisfying the following properties.

$$|f(x) - f(0) - 3x| \leq x^2 \text{ for } |x| \leq 1.$$

$$f(x+y) = f(x) + f(y) + xy(x+y) \quad \text{for all } x, y \in \mathbf{R}.$$

   (a) (5%) Find $f(0)$ and $f'(0)$.

   (b) (7%) Show that $f(x)$ is differentiable and find $f'(x)$.

   (c) (8%) Show that $f(x)$ is one-to-one and find $\left.\frac{d}{dx}f^{-1}(x)\right|_{x=\frac{10}{3}}$.

   (d) (10%) Show that for any $a < b$,

$$\int_{f(a)}^{f(b)} f^{-1}(x)\, dx = bf(b) - af(a) - \int_a^b f(x)\, dx.$$

   Find $\int_0^{\frac{10}{3}} f^{-1}(x)\, dx$.

2. (a) (10%) Find the maximum value of $f(x,y,z) = z$ on the curve of the intersection of $x+y+z=1$ and $x^2+y^2+z^2=3$.

   (b) (5%) $f(x,y,z), g(x,y,z), h(x,y,z)$ are differentiable functions. Assume that f obtains a local maximum value at (x_0, y_0, z_0) when restricted to $g(x,y,z) = c$ and $h(x,y,z) = k$. It is known that $\nabla f(x_0,y_0,z_0) = \lambda \nabla g(x_0,y_0,z_0) + \mu \nabla h(x_0,y_0,z_0)$ for some constants λ and μ. Suppose that $f(x,y,z)$ obtains new local maximum at (x_1,y_1,z_1) when restricted to $g(x,y,z) = c$ and $h(x,y,z) = k + \epsilon$ where $|\epsilon|$ is small and (x_1,y_1,z_1) is close to (x_0,y_0,z_0). Show by the linear approximation that we can approximate $f(x_1,y_1,z_1) - f(x_0,y_0,z_0)$ by $\mu \cdot \epsilon$.

   (c) (5%) Estimate, by linear approximation, the maximum value of $f(x,y,z) = z$ on the curve of the intersection of $x+y+z=1$ and $x^2+y^2+z^2=3.02$.