Notation: We denote by $\mathbf{C}$ the set of complex numbers. For any positive integer n, we denote by $\mathbf{C}^n$ the n-dimensional column vector spaces over $\mathbf{C}$; let I_n be the identity matrix in $\mathrm{M}_n(\mathbf{C})$.

Problem 1 (15pts). Let $T : \mathbf{C}^4 \to \mathbf{C}^3$ be the linear transformation defined by $T(v) = A \cdot v$, where

$$A = \begin{pmatrix} 5 & -3 & 1 & 2 \\ -1 & 3 & 3 & -2 \\ 1 & 0 & 1 & 0 \end{pmatrix} \in \mathrm{M}_{3\times 4}(\mathbf{C}).$$

(1) (5 pts) Find the rank and the nullity of T.
(2) (10pts) Find a base of $\operatorname{Ker} T$ (the kernel of T).

Problem 2 (15pts). For any complex number $a \in \mathbf{C}$, let V_a be the subspace spanned by the row vectors

$$(2, -5, a),\ (1, a, -4),\ (a, -1, -2).$$

Determine all possible values $a \in \mathbf{C}$ such that $\dim_{\mathbf{C}} V_a = 2$.

Problem 3 (25pts). Let

$$A = \begin{pmatrix} 0 & 1 & -1 \\ 2 & -1 & 2 \\ 2 & -2 & 5 \end{pmatrix}.$$

(1) (15pts) Find an invertible matrix $P \in \mathrm{M}_3(\mathbf{C})$ such that $P^{-1}AP$ is a diagonal matrix.
(2) (10pts) Find an invertible matrix $Q \in \mathrm{M}_3(\mathbf{C})$ such that

$$Q^{-1}AQ = \begin{pmatrix} 0 & 0 & -4 \\ 1 & 0 & 1 \\ 0 & 1 & 4 \end{pmatrix}.$$

Problem 4 (15pts). Let $A \in \mathrm{M}_n(\mathbf{C})$ be a Hermitian matrix $\iff A = A^*$.

(1) (5 pts) Show that $\operatorname{Ker} A \cap \operatorname{Im} A = \{0\}$.
(2) (10pts) If $A^3 = 2A^2 + 2A$, show that $A = 0$.

Problem 5 (15pts). Let $A \in \mathrm{M}_n(\mathbf{C})$ such that $A^n = 0$ but $A^{n-1} \neq 0$.

(1) (7pts) Show that there exists $v \in \mathbf{C}^n$ such that $\{v, Av, A^2v, \ldots, A^{n-1}v\}$ is a basis of $\mathbf{C}^n$.
(2) (8pts) If $B \in \mathrm{M}_n(\mathbf{C})$ such that $AB = BA$, prove that

$$B = a_0 + a_1A + a_2A^2 + \cdots a_{n-1}A^{n-1}$$

for some $a_0, \ldots, a_{n-1} \in \mathbf{C}$.

Problem 6 (15pts). Let $A, B \in \mathrm{M}_n(\mathbf{C})$. Suppose that the eigenvalues of A, B are all non-negative real numbers and that $\mathrm{null}(A) = \mathrm{null}(A^2)$ and $\mathrm{null}(B) = \mathrm{null}(B^2)$. If $A^4 = B^4$, prove that $A = B$.

(Recall that $\mathrm{null}(A)$:=the nullity of A = the dimension of the kernel of A)

試題隨卷繳回