※ 注意：請於試卷內之「非選擇題作答區」依序作答，並應註明作答之大題及小題題號。

TABLE I: The following physical constants and conversion factors may be useful in numerical calculations.

Quantity	Symbol	Value
speed of light	c	3×10^8 m/s
electron charge	e	1.6×10^{-19} C
Planck constant	h	6.63^{-34} J s
electron mass	m_e	9.1×10^{-31} kg $= 0.511$ MeV/c^2
proton mass	m_p	1.673×10^{-27} kg $= 938.27$ MeV/c^2

1. [20 points] Consider the annihilation of an electron and a positron, both at rest, into photons in quantum electrodynamics.

   (a) [10] Explain why there should be two outgoing photons at the lowest order in perturbation.

   (b) [10] What is the energy and wavelength of each outgoing photon?

2. [30 points] Suppose the Hamiltonian of a system is given by

$$\mathcal{H} = \frac{\hbar\omega}{2\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \tag{1}$$

   where $\omega > 0$.

   (a) [10] Calculate the eigenvalues and the normalized eigenvectors of $\mathcal{H}$. For the normalized eigenvectors, write them in the form of $N \begin{pmatrix} 1 \\ \alpha \end{pmatrix}$, where N and α are numbers to be derived.

   (b) [10] What is the matrix that diagonalizes $\mathcal{H}$, with the elements of the diagonalized $\mathcal{H}$ listed in the ascending order?

   (c) [10] At time $t = 0$, the system is in the $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ state, in the same basis as Eq. (1), and the energy is measured. What are the possible outcomes of the energy measurement and with what probabilities?

3. [30 points] Coherent states are a very important class of quantum states of simple harmonic oscillators. They have not only fundamental significance in modern physical studies but also various applications in quantum control and state engineering. Here, consider a simple harmonic oscillator:

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 \equiv \hbar\omega\left(a^\dagger a + \frac{1}{2}\right).$$

   (a) [10] Write down the definition for a and $a^\dagger$, and prove that $a|n\rangle = \sqrt{n}|n-1\rangle$ and $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$, where $|n\rangle$ is the eigenstate of the number operator $\hat{n} \equiv a^\dagger a$.

   (b) [10] A coherent state $|\alpha\rangle$ is defined as the eigenstate of operator a such that $a|\alpha\rangle = \alpha|\alpha\rangle$. Assume $|\alpha\rangle = \sum_n c_n |n\rangle$. Show that $\bar{n} = \langle\alpha|n|\alpha\rangle = |\alpha|^2$, and $|c_n|^2 = e^{-\bar{n}}\frac{\bar{n}^n}{n!}$.

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(c) [10] The wavefunction $\psi_\alpha(x)$ of the coherent state $|\alpha\rangle$ can be calculated by

$$a\psi_\alpha(x) = \left(\frac{x}{x_0} + \frac{x_0}{2}\frac{d}{dx}\right)\psi_\alpha(x) = \alpha\psi_\alpha(x),$$

where $x_0 = \sqrt{\frac{2\hbar}{m\omega}}$. Assuming that α is real and the solution is of the form $\psi_\alpha(x) = \mathcal{N}\exp\left[-\frac{(x-B)^2}{2W^2}\right]$ with some normalization constant $\mathcal{N}$, determine B and W, and prove $B = \langle\alpha|x|\alpha\rangle$ the expectation value of position.

4. [20 points] The Lorentz transformation is given by

$$x' = \gamma(x - vt)$$
$$y' = y$$
$$z' = z$$
$$t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

where the primed system is an inertia frame at velocity v along the x direction relative to the unprimed system, and $\gamma = 1/\sqrt{1 - v^2/c^2}$ with the speed of light c. Consider a train of proper length L moving at velocity v in the $+x$ direction. A conductor stands on the rear end of the train. When the conductor passes an observer on the platform, he immediately flashes a light to a mirror set at the front end of the train.

(a) [5] Determine the time and position viewed by the observer when the light hits the mirror.

(b) [5] Determine the time and position viewed by the conductor when the light reaches the observer after reflection .

(c) [10] Another train of the same proper length L moves on a parallel track towards the first train from ahead at velocity $-v$ viewed by the observer. From the conductor's point of view, determine the velocity of the second train and the total time duration when the two trains pass each other.

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