第一大題：選擇題 (50%)，請參照以下說明作答於「答案卡」，並詳閱答案卡上之「畫記說明」，以利電腦閱卷。

In this section, all problems are multiple choice problems. Please mark your answers on the card to be read electronically. For each of the multiple choice problems, at least one (1) and at most five (5) options are correct. Each correct choice gets one (1) point and every wrong choice gets minus one (−1) point. For example, if the correct choices are "ABCE" for a problem and your answer is "ACD", then you get $1 + (-1) + 1 + (-1) + (-1) = -1$ for that problem. If one problem is left blank, then you get zero score for that problem.

1. (5%) Which of the following is true?
   - (A) For any vector $\mathbf{v} \in \mathcal{R}^n$, $\{\mathbf{v}\}$ is linearly independent.
   - (B) If a subset of $\mathcal{R}^n$ contains more than n vectors, then it is linearly dependent.
   - (C) If every column of an $m \times n$ matrix A contains a pivot position, then the matrix equation $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b} \in \mathcal{R}^n$.
   - (D) A homogeneous equation (i.e., a system of linear equation $A\mathbf{x} = \mathbf{0}$) is always consistent.
   - (E) A homogeneous equation always has infinitely many solutions.

2. (5%) Which of the following is true?
   - (A) A matrix is invertible if and only if its reduced row echelon form is an identity matrix.
   - (B) For any two $n \times n$ matrices A and B, if $AB = I_n$, then A is invertible and $A^{-1} = B$.
   - (C) If a square matrix has a column consisting of all zeros, then it is not invertible.
   - (D) If A and B are invertible matrices, then $A + B$ is also invertible.
   - (E) If A is an $n \times n$ matrix such that $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$ in $\mathcal{R}^n$, then $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$ in $\mathcal{R}^n$.

3. (5%) Which of the following is true?
   - (A) Every function from $\mathcal{R}^n$ to $\mathcal{R}^m$, $T : \mathcal{R}^n \to \mathcal{R}^m$, has a standard matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathcal{R}^n$.
   - (B) The matrix transformation induced by an $m \times n$ matrix A (i.e., $T_A(\mathbf{x}) \triangleq A\mathbf{x}$) is a linear transformation.
   - (C) The image of the zero vector under any linear transformation is the zero vector.
   - (D) A function $T : \mathcal{R}^n \to \mathcal{R}^m$ is uniquely determined by the images of the standard vectors in its domain.
   - (E) If f is a linear transformation and $f(\mathbf{u}) = f(\mathbf{v})$, then $\mathbf{u} = \mathbf{v}$.

4. (5%) Which of the following is true?
   - (A) A linear transformation with codomain $\mathcal{R}^m$ is onto if and only if the rank of its standard matrix is m.
   - (B) A linear transformation is one-to-one if and only if its null space consists only of the zero vector.
   - (C) A linear transformation is onto if and only if the columns of its standard matrix form a generating set for its range.

(D) If the composition UT of two linear transformations $T : \mathcal{R}^n \to \mathcal{R}^m$ and $U : \mathcal{R}^p \to \mathcal{R}^q$ is defined, then $m = p$ must be true and the composition UT is also a linear transformation.

(E) For every invertible linear transformation T, the function T^{-1} is also a linear transformation.

5. (5%) Which of the following statements about the determinant is true?

(A) For any square matrix A, $\det A^T = -\det A$.

(B) The determinant of the $n \times n$ identity matrix is $(-1)^n$.

(C) The determinant of an upper triangular matrix is always equal to the product of its diagonal entries.

(D) Performing a row addition operation on a square matrix does not change its determinant.

(E) Performing a scaling operation on a square matrix does not change its determinant.

6. (5%) Which of the following is true?

(A) Every nonzero subspace of $\mathcal{R}^n$ has a unique basis.

(B) Every subspace of $\mathcal{R}^n$ has a basis composed of standard vectors.

(C) The column space of an $m \times n$ matrix is contained in $\mathcal{R}^m$.

(D) If V is a subspace of dimension k, then every generating set for V contains exactly k vectors.

(E) The pivot columns of a $m \times n$ matrix A form a basis for the column space of A.

7. (5%) Which of the following is an eigenvector of $A = \begin{bmatrix} 4 & -2 & -2 \\ 1 & 1 & -1 \\ -1 & 1 & 3 \end{bmatrix}$?

(A) $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$; (B) $\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$; (C) $\mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$; (D) $\mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$;

(E) None of the above.

8. (5%) Which of the following is a linear operator?

(A) $T : \mathcal{R} \to \mathcal{R}$ where $T(x) = 4x + 3$ for any $x \in \mathcal{R}$.

(B) $T : \mathcal{R}^2 \to \mathcal{R}^2$ where $T(\mathbf{x}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ for any vector $\mathbf{x} \in \mathcal{R}^2$.

(C) $T : \mathcal{P} \to \mathcal{P}$ where $T(f(x)) = f(x)(x^2 + 1)$ for any polynomial $f(x) \in \mathcal{P}$.

(D) $T : \mathsf{C}^\infty(\mathcal{R}) \to \mathsf{C}^\infty(\mathcal{R})$ where $T(f(x)) = f'(x) + f(x)$ for any differentiable function $f(x) \in \mathsf{C}(\mathcal{R})$.

(E) None of the above.

9. (5%) Which of the following sets is an orthonormal basis for the designated vector space?

(A) $\left\{ \frac{1}{5} \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \frac{1}{5} \begin{bmatrix} 4 \\ -3 \end{bmatrix} \right\}$, for $\mathcal{R}^2$ with $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \mathbf{y}$.

(B) $\{\cos x, \sin x\}$, for $\mathsf{C}[0, 2\pi]$ with $\langle f, g \rangle = \int_0^{2\pi} f(x)g(x)dx$.

(C) $\{1, x, x^2\}$ for $\mathcal{P}_3$ with $\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)dx$.

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(D) $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \frac{1}{2}\begin{bmatrix}0\\1\\-1\end{bmatrix}, \frac{1}{2\sqrt{2}}\begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$ for $\mathcal{R}^3$ with $\langle \mathbf{x}, \mathbf{y}\rangle = \mathbf{x}^T M \mathbf{y}$ where $M = \begin{bmatrix}1&0&0\\0&3&1\\0&1&3\end{bmatrix}$.

(E) None of the above.

10. (5%) Which of the following is a vector space?

(A) $\mathcal{P} = \{p \mid p \text{ is a polynomial in } x \text{ of any order}\}$

(B) $\mathcal{P}_n = \{p \mid p \text{ is a polynomial in } x \text{ of exactly } n\text{th order}\}$

(C) $\{T \mid T : \mathcal{R}^n \to \mathcal{R}^m, T \text{ is a linear transformation}\}$

(D) The set of all $m \times n$ matrices of which the sum of all entries is zero, i.e., $\left\{A \in \mathcal{R}^{m\times n} \mid \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij} = 0\right\}$.

(E) The set of all $n \times n$ square matrices with trace 1, i.e., $\{A \in \mathcal{R}^{n\times n} \mid \sum_{i=1}^{m} a_{ii} = 1\}$.

第二大題：非選擇題 (50%)，請於試卷內之「非選擇題作答區」標明題號依序作答。答案請標示清楚，答案完全正確才給分。

11. (30%) A study show that in an exam the time required to come up with the correct answer for each question is a continuous memoryless random variable with the expected value equal to the difficulty level of the question. The time to answer each question is independent of each other.

(a) (6%) The first and second questions both have difficulty level 2. Denote T as the random variable representing the total time required to come up with the correct answers for both questions. Write down the PDF of T.

(b) (6%) The third question has difficulty level 3 while the fourth question has difficulty level 5. Denote U as the random variable representing the difference in time to come up with the correct answers for the two questions (time for the fourth minus time for the third). Write down the PDF of U.

(c) (6%) The fifth question has difficulty level 7 while the sixth question has difficulty level 8. Due to time constraint, you have time to write only one question. What is the probability that the time to finish the fifth question is shorter than the time to finish the sixth question?

(d) (6%) In another exam, there are so many questions such that it seems impossible to finish all of them. Fortunately, all questions in the exam have the same difficulty level 2. Denote V as the number of questions answered correctly for the exam within the total allowable time of 100 (time units). Write down the PMF of V.

(e) (6%) 50 students participate in an exam and all of them start with the first question at the same time. The difficulty level of the first question is 2. What is the probability that the question is answered correctly within 2 time units by any of the 50 students?

12. (12%) A bunch of sensors are dropped from an aircraft to a forest for monitoring the occurrence of the wildfire. It is known that the wildfire can occur at random location (X, Y) in the forest with the joint PDF of X and Y as follows:

$$f_{X,Y}(x,y) = \frac{ab}{4}e^{-a|x|-b|y|}, \quad -\infty < x, y < \infty \text{ and } a, b > 0.$$

It is found that dropped sensors are distributed over a square region with side length $2D$ centered at $(0,0)$. Within the distribution region, the number of sensors in any sub-region of area A follows the Poisson (ρA) distribution, where ρ is the distribution density (sensor/unit area). When a sensor is dropped to the ground, it has a probability q, $0 < q < 1$, of malfunction that renders it totally useless. If a sensor is *functional*, it has a circular detection area πR^2 such that it can successfully detect the wildfire within a radius R with probability p, $0 < p < 1$, and not detect anything with probability $1 - p$. Assume for the typical scenario $D > R$.

(a) (6%) Let N be the number of *functional* sensors in any sub-region of area A. Write down the PMF of N.

(b) (6%) Define the *core* of the distribution region as a square with side length $2(D - R)$ centered at $(0,0)$, where no boundary effect of sensor distribution needs to be considered. If a wildfire occurs within the *core*, find the probability that it is successfully detected.

13. (8%) Alice and Bob are playing a game against the house (host of the game). A circle is drawn on the wooden wall, with the inner region (disk) assigned to Alice and the outer region assigned to Bob. The center is labeled as the origin (0, 0) and Alice and Bob take turns to throw darts to the circle. When Alice throws a dart the coordinates of the landing point are a Gaussian pair of independent random variables each with zero mean and variance 1. When Bob throws the dart the coordinates are also a Gaussian independent pair but with zero mean and variance 4. Whenever Alice's dart lands on Bob's region, she must pay \$2 dollar to the house; whenever Bob's dart lands on Alice's region, he must pay \$1 dollar to the house. Find the disk radius that minimizes the players' average expense.

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