(1) (15%) Let $V = \mathbb{R}^6$. Let W_1 be the subspace of V spanned by

$$(1,2,3,4,5,6),(3,4,6,7,9,10),(0,1,0,2,0,3),(1,-2,3,-4,5,-6),$$

and W_2 be the subspace of V spanned by

$$(1,1,1,2,2,3),(-2,0,-1,0,1,2),(1,0,1,0,2,0),(0,0,1,0,-2,-2).$$

Find the dimension of the subspace $W_1 \cap W_2$ and find a basis for this subspace.

(2) (15%) Let

$$C = \begin{bmatrix} -x & 1 & 3 & 1 & 2 \\ -2 & 0 & x & 2 & 2 \\ x & 0 & -2 & -3 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & x & -2 \end{bmatrix}.$$

Find an integer x such that all entries of the inverse of C are integers. For such x, find C^{-1}.

(3) (15%) Let V be the vector space of all $n \times n$ matrices over F. Let T be the linear operator on V defined by $T(A) = A^t$. Test T for diagonalizability, and if T is diagonalizable, find a basis for V such that the matrix representation of T is diagonal.

(4) (15%) Let V and W be F-vector spaces, and V^* and W^* be the dual space of V and W, respectively. Let $T : V \to W$ be a linear transformation. Define $T^* : W^* \to V^*$ by $T^*(f) = f \circ T$ for all $f \in W^*$. Show that T is onto if and only if T^* is one to one.

(5) (10%) Let A and B be $n \times n$ matrices over a field F. Show that if A is invertible, there are at most n scalars c in F such that $cA + B$ is not invertible.

(6) (15%)

(a) Let S and T be linear operators on a finite-dimensional vector space. If $p(t)$ is a polynomial such that $p(ST) = 0$, and if $q(t) = tp(t)$, show that $q(TS) = 0$.

(b) What is the relation between the minimal polynomials of ST and TS.

(7) (15%) Let V be a vector space with a basis $\{u_1, u_2, \ldots, u_n\}$. Let $\langle\ ,\ \rangle$ be an inner product on V. If $c_1, c_2, \ldots, c_n$ are any n scalars, show that there is exactly one vector v in V such that $\langle v, u_j \rangle = c_j$, $j = 1, 2, \ldots, n$.

試題隨卷繳回