2019 Graduate Entrance Exam- Modern Physics

1. (20 points) Relative to an observer at rest in the lab, $4N_0$ ($N_0 \gg 1$) particles are uniformly distributed and confined to move at a constant speed v on a square track $ABCD$ of side L_0 resting on the x-y plane of the lab (Fig. 1). A primed frame is moving at the velocity v along the positive x-axis of the lab frame, and the two coordinate systems coincide at $t = t' = 0$. The Lorentz transformation between the two frames is

$$\begin{aligned} x' &= \gamma(x - vt), \\ t' &= \gamma\left(t - \frac{v}{c^2}x\right), \\ y' &= y, \quad z' = z, \\ \gamma &\equiv \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}, \end{aligned}$$

where c is the speed of light in vacuum. In answering the following questions, you can freely quote the familiar concept of "length contraction," "time dilation," etc. without resorting to the above formulas if you are confident with the physics involved. But if in doubt, you are welcome to apply the above formulas to help yourself get around any difficulties you might encounter. In other words, *these formulas are provided for your convenience only*, and you are *not* required to use them in answering the following questions.

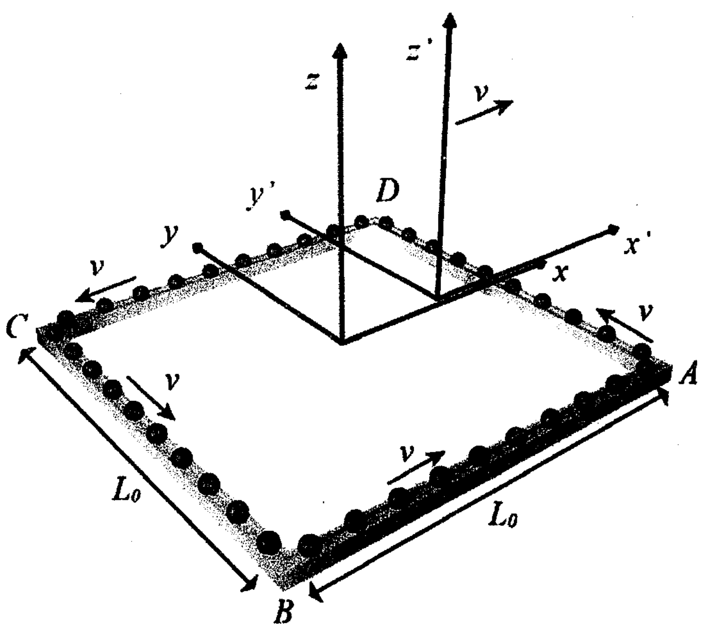

Figure 1: Fig. 1: Uniformly distributed particles are confined to move at a speed v on a square track of side L_0 that is at rest in the lab.

In what follows, physical quantities measured in the primed frame are denoted with a prime (′). For instance, the length of the side $\overline{AB}$ observed by the primed observer is denoted by L'_{AB}, and the number of particles contained within the two ends A and B is denoted by N'_{AB}. We clearly have $L_{AB} = L_{BC} = L_0$, and $N_{AB} = N_{BC} = N_0$. For each of the following statements, please answer true or false, and then provide a physics explanation. You earn credits only if both your answer and the explanation are correct.

(a) (5 points) $L'_{AB} < L_0$.

(b) (5 points) $\frac{L'_{AB}}{N'_{AB}} < \frac{L_0}{N_0}$.

(c) (5 points) $N'_{BC} = N_0$.

(d) (5 points) $N'_{CD} = \left(1 + \frac{v^2}{c^2}\right) N_0$.

2. (30 points) For a particle of mass M moving in a static three-dimensional potential $V(\vec{r})$, the time-dependent Schrödinger equation for the wavefunction $\psi(\vec{r}, t)$ reads

$$-\frac{\hbar^2}{2M}\nabla^2\psi + V(\vec{r})\psi = i\hbar\frac{\partial\psi}{\partial t}. \tag{1}$$

(a) (5 points) If we define the local probabilistic density ρ and the local probabilistic flux $\vec{j}$ as

$$\rho \equiv \psi^*\psi,$$
$$\vec{j} \equiv \frac{\hbar}{2Mi}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right),$$

respectively, then please use Eqn.(1) to show that

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\vec{j} = 0.$$

In the above, ψ^* is the complex conjugate of ψ.

(b) (8 points) Please show that

$$\frac{d}{dt}\int\left(M\vec{j}\right)d^3r = -\int(\nabla V)\,\rho d^3r,$$

which can be viewed as a quantum mechanical interpretation of Newton's second law of motion.

(c) (5 points) If we write $\psi \equiv Ae^{iS}$, where A and S are real-valued functions, then please show that

$$\vec{j} = \frac{\hbar}{M}A^2\nabla S.$$

(d) (5 points) We now restrict ourselves to the case when V is a spherically symmetric central field, i.e., V depends only the radial distance r. For an energy eigenstate with the energy E, one traditionally adopts spherical coordinates (r, θ, ϕ) and uses "separation of variables" to express ψ as

$$\psi = R(r)\,\Theta(\theta)\,e^{im\phi}e^{-i\frac{E}{\hbar}t},$$

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where R and Θ are some real-valued functions each satisfying a certain second order differential equation *not of our concern here*. In the above, m is some integer. Suppose we define the local orbital angular momentum density $\vec{J}_\rho$ as

$$\vec{J}_\rho \equiv \vec{r} \times M\vec{j},$$

please show that the z-component of $\vec{J}_\rho$ is $\rho m\hbar$. Note: The gradient operator expressed in spherical coordinates is

$$\nabla = \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{\partial}{r\partial\theta} + \hat{e}_\phi \frac{\partial}{r \sin\theta \partial\phi}.$$

(e) (7 points) Restricting ourselves to the one-dimensional case with a potential

$$V(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ \infty & \text{if } x > 0. \end{cases}$$

Suppose initially we have a wavepacket of form

$$\psi(x, t = 0) = e^{i20x} e^{-(x-5)^2/4}$$

incident from the left. Please draw a schematic plot of $\rho(x)$ when

1. (2 points) $t = 0$,
2. (5 points) and when the peak of the wavepacket approximately reaches $x = 0$, paying special attention to x near the origin.

3. (25 points) It is said that the famous sodium D-lines (D_1 and D_2) arise from the spin-orbit interaction. D_1 is due to the $3^2P_{1/2} \to 3^2S_{1/2}$ transition and D_2 is due to the $3^2P_{3/2} \to 3^2S_{1/2}$ transition.

   (a) (10 points) What is the so-called spin-orbit interaction?

   (b) (5 points) Which state, $3^2P_{1/2}$ or $3^2P_{3/2}$, has higher energy? Why?

   (c) (10 points) What happens to the D-Lines if the sodium atom is placed in an external magnetic field? (You need to explain what happens to D_1 and D_2 separately.)

4. (20 points) Consider N free electrons (i.e. interactions between electrons are neglected) in a 3-dimensional cubical box of side L. The mass of electron is m.

   (a) (5 points) If $N = 1$, what are the allowed energies E for this electron ?

   (b) (15 points) When N is very large, the Fermi energy E_F of the system is related to N by the formula $E_F = cN^a$, where c and a are constants. What are c and a?

5. (5 points) Give the numerical values (in appropriate units) for

   (a) (2 points) lifetime of free neutron

   (b) (3 points) mass of neutral pion