※ 注意：請於試卷上「非選擇題作答區」標明大題及小題題號，並依序作答。

本試卷禁止使用計算器.

Notation: Let $\mathbf{R}$ and $\mathbf{C}$ denote the set of real and complex numbers respectively. For $F = \mathbf{R}$ or $\mathbf{C}$ and any positive integer n, I_n is the identity matrix and 0_n is the zero matrix in $\mathrm{M}_n(F)$. Let F^n denote the column vector space of dimension n over F. If $A \in \mathrm{M}_{m\times n}(F)$, $A^{\mathrm{t}} \in \mathrm{M}_{n\times m}(F)$ denote the transpose of A. For $x = (x_1, \ldots, x_n)^{\mathrm{t}} \in \mathbf{C}^n$, recall that the norm $\|x\|$ of x is defined by

$$\|x\| := \sqrt{|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2}.$$

Problem 1 (20pts). Let $T : \mathbf{R}^5 \to \mathbf{R}^4$ be the linear transformation defined by $T(v) = A \cdot v$ with the matrix

$$A = \begin{pmatrix} 2 & 3 & 1 & 4 & -9 \\ 1 & 1 & 1 & 1 & -3 \\ 1 & 1 & 1 & 2 & -5 \\ 2 & 2 & 2 & 3 & -8 \end{pmatrix}.$$

(1) (5pts) Find the dimension of the image of T.
(2) (10pts) Find bases of the kernel of T and the image of T.
(3) (5pts) Find some $x \in \mathbf{R}^5$ such that $T(x) = (17, 6, 8, 14)^{\mathrm{t}}$.

Problem 2 (10pts). Let W be the subspace of $\mathbf{R}^4$ defined by

$$W = \left\{(x_1, x_2, x_3, x_4)^{\mathrm{t}} \in \mathbf{R}^4 \mid x_1 + x_2 + x_3 - x_4 = 0,\ x_1 - 3x_2 + 2x_3 = 0\right\}.$$

Let $v = (2, 1, 0, 1)^{\mathrm{t}}$. Find the minimal distance from v to W, namely $\min_{w\in W}\|v - w\|$.

Problem 3 (25 pts). Let

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \in \mathrm{M}_3(\mathbf{R})$$

be a symmetric matrix.

(1) (5pts) Find the eigenvalues of A.
(2) (20pts) Find a matrix P such that $PP^{\mathrm{t}} = I_n$ and $P^{-1}AP$ is a diagonal matrix.

Problem 4 (15pts). Let $A, B \in \mathrm{M}_n(\mathbf{C})$ such that $\operatorname{rank} A = \operatorname{rank} B$ and $A^2B = A$. Prove that $B^2A = B$.

Problem 5 (15pts). Determine all of the matrices $X \in M_2(\mathbf{R})$ that satisfy the equation

$$X^3 - 3X^2 = \begin{pmatrix} 1 & -3 \\ 1 & -3 \end{pmatrix}.$$

Problem 6 (15pts). Let $A \in \mathrm{M}_n(\mathbf{C})$ with $\det A = 1$ and $\|Ax\| \le \|x\|$ for any $x \in \mathbf{C}^n$. Prove that $AA^* = I_n$.

試題隨卷繳回