國立臺灣大學 112 學年度碩士班招生考試試題

1. (10 pts) Let $\gamma(s) : [0, L] \to \mathbf{R}^2$ be a regular curve with $||\gamma'(s)|| = 1$. Assume that its curvature satisfies $\kappa(s) > 0$. Let $l > 0$ and define $\alpha(s) = \gamma(s) + (l - s)\gamma'(s)$, for $0 < t < l$. Show that α is a regular curve, and compute its curvature.

2. (15 pts) For a space curve in $\mathbf{R}^3$, we use the convention of the Frenet frames that satisfy

$$T'(s) = \kappa(s)N(s), N'(s) = -\kappa(s)T(s) + \tau(s)B(s), B'(s) = -\tau(s)N(s).$$

   Determine whether or not there exists a smooth unit-speed curve $\beta : (0, 1) \to \mathbf{R}^3$ such that $\kappa(s) > 0$ for all $s \in (0, 1)$ and such that the unit binormal satisfies $B''(s) = s^3T(s) + 4B(s)$.

3. Let S be a surface of revolution with parametrization $\psi(u, v) = (\phi(u)\cos v, \phi(u)\sin v, u)$ where $\phi(u)$ is smooth and positive..
   (a) (10 pts) Find the principal directions, mean curvature and compute the Gaussian curvature $K(u, v)$ for any point $p = \psi(u, v)$.
   (b) (5 pts) Show that any meridian curve ($v = v_0$) is a geodesic.
   (c) (5 pts) Show that a parallel curve ($u = u_0$) is a geodesic if and only if $\phi'(u_0) = 0$.

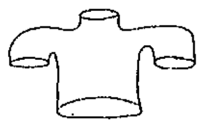
4. Let $S^2 = \{(x, y, z) \in \mathbf{R}^3 : x^2 + y^2 + z^2 = 1\}$ be the unit sphere. Let

$$\omega = \begin{cases} \frac{dy\wedge dz}{x} & \text{if x} \neq 0; \\ \frac{dz\wedge dx}{y} & \text{if y} \neq 0; \\ \frac{dx\wedge dy}{z} & \text{if x} \neq 0. \end{cases} \tag{1}$$

   (a) (7 pts) Show that ω is a well defined two form on S^2.
   (b) (7 pts) Find $\int_{S^2} \omega$
   (c) (6 pts) Is ω exact? Why?

5. Let S be a smooth, compact, orientable surface in $\mathbf{R}^3$.
   (a) (10 pts)) Show that the Gauss map $N : S \to S^2$ is surjective.
   (b) (10 pts)) Let $K(p)$ be the Gaussian curvature of S at p and $K_+(p) = \max\{0, K(p)\}$.
   Show that $\int_S K_+ dA \geq 4\pi$.

6. (15 pts) What is the integral of the Gaussian curvature of the following surface? You may assume that the boundary curves are geodesics.



試題隨卷繳回