國立臺灣大學111學年度碩士班招生考試試題

1. (20 pts) Let γ be a regular curve in $\mathbf{R}^2$ parametrized by arc-length such that $||\gamma(s)|| \leq 1$ for all s. Suppose that there is a point s_0 where $||\gamma(s_0)|| = 1$. Prove that the curvature at that point satisfies $|\kappa(s_0)| \geq 1$.

2. (20 pts) Prove that a torus in $\mathbf{R}^3$ cant have Gauss curvature which is everywhere ≥ 0.

3. (20 pts) Let γ be a regular curve on a surface regular surface M with Gaussian curvature $K > 0$ at $p \in \gamma$. Show that the curvature κ of γ at p satisfies $\kappa \geq \min\{|k_1|, |k_2|\}$ where k_1 and k_2 are the principal curvatures of M at p.

4. Let S be a regular surface parametrized by $\sigma(u, v)$ and S^t be the parallel surface at distance t from S which can be parametrized by $\sigma^t(u, v) = \sigma(u, v) + tN(\sigma(u, v))$ where N is the unit normal vector field at $\sigma(u, v)$.

   (a) (10 pts) Show that $N_u \times N_v = K\sigma_u \times \sigma_v$ and $\frac{\sigma_v \times N_u + N_v \times \sigma_u}{2} = H\sigma_u \times \sigma_v$ where K and H are the Gaussian and mean curvatures of S.

   (b) (10 pts) Show that S^t is a parametrized surface if $|t| < min\{\frac{1}{|\kappa_1|}, \frac{1}{|\kappa_2|}\}$ and N is also the natural unit normal for S^t.

   (c) (10 pts) Show that
   $$\sigma_u^t \times \sigma_v^t = (1 - 2tH + t^2K)\sigma_u \times \sigma_v.$$

   (d) (10 pts) Show that the Gaussian and mean curvature of S^t are $\frac{K}{1-2tH+t^2K}$ and $\frac{H-tK}{1-2tH+t^2K}$, respectively, where K and H are the Gaussian and mean curvatures of S. (Hint: Do not compute it by brutal force. Use Part (a) instead.)