※ 注意：請於試卷內之「非選擇題作答區」依序作答，並應註明作答之大題及小題題號。

※ Please show the detailed calculation process for the questions whenever necessary.
※ If the answers are with decimal numbers, please round to the second decimal place, e.g., 99.37 or 2.43%.

1. (30%) Consider a down-and-in call option whose payoff at maturity T is

$$c_T = \max(S_T - K, 0)$$

only if the stock price S_t has reached the barrier level B before the maturity date, where S_t denotes the stock price at time $t \in [0, T]$, K denotes the strike price, and the barrier level B is assumed to be lower than $\min(S_0, K)$.

Under the Black-Scholes framework, its value function today ($t = 0$) is

$$c_0 = S_0 \exp(-qT)\left(\frac{B}{S_0}\right)^{2\lambda} N(y) - K\exp(-rT)\left(\frac{B}{S_0}\right)^{2\lambda-2} N(y - \sigma\sqrt{T}),$$

where $\lambda = \frac{r-q+\frac{\sigma^2}{2}}{\sigma^2}$, $y = \frac{\ln\left(\frac{B^2}{S_0 K}\right)+\left(r-q+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$, r is the risk-free interest rate, q is the dividend yield, σ is the stock price volatility, $N(\cdot)$ is the cumulative distribution function of the standard normal distribution defined as

$$N(d) = \int_{-\infty}^{d} n(x)dx = \int_{-\infty}^{d} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx,$$

and $n(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ is the probability density function of the standard normal distribution.

(a) (10%) Prove that $S_0 \exp(-qT)\left(\frac{B}{S_0}\right)^{2\lambda} n(y) = K\exp(-rT)\left(\frac{B}{S_0}\right)^{2\lambda-2} n\left(y - \sigma\sqrt{T}\right)$.

(b) (10%) Derive $\frac{\partial c_0}{\partial S_0}$. Express $\frac{\partial c_0}{\partial S_0}$ as $\alpha N(y) + \beta N(y - \sigma\sqrt{T})$ under the assumption of $\lambda = 0.5$. What are α and β? Note that $N(y)$, $N(y - \sigma\sqrt{T})$, $n(y)$, and $n\left(y - \sigma\sqrt{T}\right)$ are not allowed to appear in α and β.

(c) (10%) Derive $\frac{\partial^2 c_0}{\partial {S_0}^2}$. Express $\frac{\partial^2 c_0}{\partial {S_0}^2}$ as $\gamma n\left(y - \sigma\sqrt{T}\right)$ under the assumption of $\lambda = 0.5$. What is γ? Note that $N(y)$, $N(y - \sigma\sqrt{T})$, $n(y)$, and $n\left(y - \sigma\sqrt{T}\right)$ are

見背面

國立臺灣大學 111 學年度碩士班招生考試試題

not allowed to appear in γ.

2. (7%) A company begins advertising a new product and finds that after t weeks the product is gaining customer recognition at the rate of $t^3 \ln t$ thousand customers per week (for $t \geq 1$). Find the total gain in recognition from the end of week 1 to the end of week 4.

3. (13%) Find the least squares approximation $h(x) = a_0 + a_1 x$ for $f(x) = e^{-x}$, $0 \leq x \leq 1$. [Hint: Solve a_0 and a_1 through minimizing the integral of the squared difference between $f(x)$ and $h(x)$.]

4. (10%) Determine the open intervals on which the graph of the function is concave upward or concave downward:

$$f(x) = \frac{6}{x^2+3}$$

5. (10%) Find the area of the region bounded by the graph of

$$y = x^2 - 3x - 4$$

and the x-axis.

6. (10%) Use a double integral to find the area of the region bounded by the graphs of $y = x^2$ and $y = x^3$.

7. (10%) Estimate the point of intersection of the graphs of $y = e^{-x^2}$ and $y = x$.

8. (10%) Find the general solution of $x\frac{dy}{dx} - 2y = x^2$.

試題隨卷繳回