※ 禁止使用計算機

注意事項：

i. 問題 1. 至 4. 皆假設 over $\mathbb{R}$；問題 5. over $\mathbb{C}$。

ii. 答題引述任何定理時，必須敍述清楚。

iii. 不得使用計算器或其他 3C 產品。

iv. 請於答題本「非選擇題作答區」標明題號作答。

記號

$\mathbb{R}$: real number; $\mathbb{C}$: complex number; $\mathbb{R}^n$: n dimensional Euclidean space.

$\mathbf{M}_n$: space of $n \times n$ matrices with entries in $\mathbb{R}$ ($\mathbf{M}_n^{\mathbb{C}}$: entries in $\mathbb{C}$).

$\mathbf{P}_n$: vector (linear) space of real polynomials of degree less than or equal to n

試題

1. [20%] Fixed $a \in \mathbb{R}$. Define a function $F : \mathbf{P}_n \to \mathbb{R}^{n+1}$ by
$$f(x) \mapsto [f(a)\ f'(a)\ f''(a)\ \cdots\ f^{(n)}(a)]^T.$$
   a. Show that F is a linear transformation.
   b. What is the condition for F to be a linear isomorphism? Determine the polynomial $F^{-1}([\alpha_0\ \alpha_1\ \alpha_2\ \cdots\ \alpha_n]^T)$ when it is possible.

2. [20%] $A \in \mathbf{M}_3$. Suppose $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are linear independent vectors and
$$\begin{cases} A\mathbf{u} = & 2\mathbf{v} + 2\mathbf{w} \\ A\mathbf{v} = & \mathbf{u} + \mathbf{v} + 2\mathbf{w} \\ A\mathbf{w} = & -\mathbf{u} + \mathbf{v} + \mathbf{w} \end{cases}$$
Show that A is diagonalizable and determine the eigenbasis in term of $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$.

3. [20%] Determine the Jordan form of A and the corresponding Jordan basis.
$$A = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

4. [20%] Show that if $A = [a_{ij}] \in \mathbf{M}_n$ is positive definite, then all diagonal elements a_{ii} are positive and $\max\limits_{1 \le i,j \le n} |a_{ij}|$ is on the diagonal. How about the converse statement?

5. [20%] For $A \in \mathbf{M}_n^{\mathbb{C}}$, we assume the fact that if A is normal (i.e. $AA^* = A^*A$, where $A^* = \overline{A}^T$ is the adjoint of A) then A is unitarily diagonalizable. Suppose $\lambda_1, \lambda_2, \cdots, \lambda_k$ are eigenvalues of A, where λ_i are distinct. Prove the following statements:
   a. If A is normal, there are $P_1, P_2, \cdots, P_k \in \mathbf{M}_n^{\mathbb{C}}$ such that
$$A = \lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_k P_k$$
   and for any polynomial function $f(t)$,
$$f(A) = f(\lambda_1)P_1 + f(\lambda_2)P_2 + \cdots + f(\lambda_k)P_k.$$
   b. A is normal if and only if $A^* = p(A)$ for certain polynomial function $p(t)$.

試題隨卷繳回