※ 注意：請於試卷內之「非選擇題作答區」標明題號依序作答。

(1) [5 + 10 + 10 分] A function $\mathrm{d} : M \times M \to [0, \infty)$ is called an *ultrametric* if

- $\mathrm{d}(x, y) = 0$ if and only if $x = y$;
- $\mathrm{d}(x, y) = \mathrm{d}(y, x)$;
- $\mathrm{d}(x, z) \le \max\{\mathrm{d}(x, y), \mathrm{d}(y, z)\}$

for any $x, y, z \in M$. We call (M, d) an ultrametric space.

(a) Show that an ultrametric space, (M, d), is a metric space.

(b) In an ultrametric space, show that "all triangles are isosceles[1]".

(c) Prove that an ultrametric space must be totally disconnected[2].

(2) [15 + 15 分] For the following series, determine whether it converges or diverges. If the series converges, determine whether the convergence is absolutely or conditionally convergent. Justify your answer.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n+1)\log(n+1)}$, where log means natural logarithm.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (n+1)}{\sqrt[n]{n}}$.

(3) [5 + 10 + 10 分] Consider

$$f_n(x) = \frac{\sin(nx)}{2nx} \quad \text{for } x > 0 .$$

(a) For every $x > 0$, determine $\lim_{n\to\infty} f_n(x)$.

Let $f(x)$ be the function given by the pointwise limit you found in part (a).

(b) Fix a positive number ε. Does $\{f_n(x)\}_{n\in\mathbb{N}}$ converges to $f(x)$ *uniformly* over $[\varepsilon, \infty)$? Give your reason.

(c) Does $\{f_n(x)\}_{n\in\mathbb{N}}$ converges to $f(x)$ *uniformly* over $(0, \infty)$? Give your reason.

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[1] An isosceles triangle is a triangle that has two sides of equal length.

[2] A metric space M is totally disconnected if for every $x \in M$ and any $\varepsilon > 0$, there exists a subset $U \subset$ which is both open and closed, and which is contained in the open ball of radius ε centered at x.

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(4) [20 分] Let F be a smooth map from $\mathbb{R}^3$ to $\mathbb{R}^3$. Denote by (x_1, x_2, x_3) the coordinate for the domain $\mathbb{R}^3$. Denote the origin $(0,0,0)$ by $\mathbf{O}$. Suppose that

$$\det(DF|_{(x_1,x_2,x_3)}) = 0 \quad \text{for every} \quad (x_1,x_2,x_3) \in \mathbb{R}^3 ,$$

$$F(\mathbf{O}) = \mathbf{O} , \quad \left.\frac{\partial F}{\partial x_1}\right|_{\mathbf{O}} = (0,4,0) \quad \text{and} \quad \left.\frac{\partial F}{\partial x_2}\right|_{\mathbf{O}} = (3,0,0) .$$

Prove that there exist

- open neighborhoods $U, \tilde{U}$ of $\mathbf{O}$ in (the domain) $\mathbb{R}^3$, and a diffeomorphism $\varphi : \tilde{U} \to U$ which maps $\mathbf{O}$ to $\mathbf{O}$,
- open neighborhoods $V, \tilde{V}$ of $\mathbf{O}$ in (the target) $\mathbb{R}^3$, and a diffeomorphism $\psi : \tilde{V} \to V$ which maps $\mathbf{O}$ to $\mathbf{O}$,

such that $\psi^{-1} \circ F \circ \varphi$ as a map from $\tilde{U} \subset \mathbb{R}^3$ to $\tilde{V} \subset \mathbb{R}^3$ is

$$(\psi^{-1} \circ F \circ \varphi)(y_1, y_2, y_3) = (y_1, y_2, 0) .$$

What follows is the diagram for your reference.

$$\begin{array}{ccccc} \mathbb{R}^3 \supset & U & \xrightarrow{F} & V & \subset \mathbb{R}^3 \\ & \varphi \uparrow & & \uparrow \psi & \\ \mathbb{R}^3 \supset & \tilde{U} & \dashrightarrow & \tilde{V} & \subset \mathbb{R}^3 \end{array}$$