國立臺灣大學 102 學年度碩士班招生考試試題

1. (20 pts) Evaluate

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-x^2}}{1+x^2+2xy+y^2}\,dxdy.$$

2. (30 pts) Let $g(x,y)$ be a function satisfying $-1 < g(x,y) < 1$ and

$$\ln\left(\frac{1+g(x,y)}{1-g(x,y)}\right)+2y\tan^{-1}(yg(x,y))=2(y^2+1)x$$

for $-\infty < x < \infty, y > 1$, where $\tan^{-1} = \arctan$ maps $(-\infty,\infty)$ to $(-\pi/2,\pi/2)$.
(a) Show that $g(x,y)$ is increasing in x.
(b) Find the limit function $\bar{g}(x) = \lim_{y\to\infty} g(x,y)$.
(c) Show that $g(x,y)$ is a differentiable function in (x,y).
(d) Find $\lim_{y\to\infty}\frac{\partial}{\partial x}g(x,y)$.

3. (25 pts) Let

$$f(x)=\begin{cases}-1, & -\pi\le x<0,\\ 1, & 0\le x\le \pi.\end{cases}$$

(a) Find the Fourier series of $f(x)$ on $[-\pi,\pi]$.
(b) Use (a) to find the sum of the series

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots.$$

4. (25 pts) Let $f_1(x)=3$ and $f_{n+1}(x)=\frac{1}{2}\left(f_n(x)+\frac{e^x}{f_n(x)}\right)$ for $n=1,2,3,\cdots$.
(a) Show that $f_{n+1}(x) < f_n(x)$ for $0\le x\le 1$.
(b) Show that $f(x)=\lim_{n\to\infty} f_n(x)$ exists and find $f(x)$ for $0\le x\le 1$.
(c) Does $\{f_n\}$ converge uniformly to $f(x)$ on $[0,1]$?

試題隨卷繳回