1. (13%) Consider a system of ODE $\begin{cases} y_1' = -3y_1 + y_2 - 6e^{-2t} \\ y_2' = y_1 - 3y_2 + 2e^{-2t} \end{cases}$.
Find the eigenvalues and the eigenvectors of this system. Check the orthogonality and the linear independence of the eigenvectors. Determine the fundamental matrix $\Omega(t)$ and its inverse $\Omega^{-1}(t)$. Write the general solution of the system. Determine a unique solution if $y_1(0)=1$ and $y_2(0)=-1$.

2. (27%) Consider $ty''-ty'-y=0$ subjected to $y(0)=0$ and $y'(0)=3$.
   (1) Let $F(s)=L[f(t)]$ denote the Laplace transform of a time function $f(t)$. Show that $dF/ds=-L[tf(t)]$. Determine d^nF/ds^n.
   (2) Apply Laplace transform to solve the given ODE.
   (3) Obtain a series solution around $t=0$ for the same ODE (explain if any particular solution format is used).

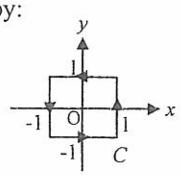
3.
   (1) (8%) Assuming a function $f(x)$ is defined as $f(x)=-x \quad -1\le x\le 1$, derive and plot its Fourier series and Fourier-Legendre series representations. In addition, roughly sketch the first and the tenth partial sum of both series.
   (2) (2%) In general, under what conditions a function equals its Fourier series?

4. Consider the one-dimensional heat conduction problem $\dfrac{\partial u}{\partial t}=k\dfrac{\partial^2 u}{\partial x^2} \quad 0<x<\infty,\ t>0$ with boundary condition $u(0,t)=0$ and initial condition $u(x,0)=f(x)$.
   (1) (8%) Derive the general solution $u(x,t)$.
   (2) (2%) Explain why the solution is in the integral form but not in the series form.

5.
   (1) (2%) Show Laplace's equation in polar coordinates, and briefly explain what a Dirichlet problem is.
   (2) (8%) Solve the following problem:

$$\nabla^2 u(r,\theta)=0 \quad \text{for } 0\le r\le 1,\quad -\pi\le\theta\le\pi$$
$$u(1,\theta)=\theta^2 \quad \text{for } -\pi\le\theta\le\pi$$

6. Write down the answers to the following questions. **(Derivations are not required.)**
   (1) (3%) Let the instantaneous position of a particle in motion be given by the following vector function: $\underline{\mathbf{r}}(t)=t\,\underline{\mathbf{i}}+t\sin t\,\underline{\mathbf{j}}+t\cos t\,\underline{\mathbf{k}}$. What is the tangential component of the particle's acceleration?
   (2) (3%) In the above problem, what is the normal component of the particle's acceleration?
   (3) (3%) Find the angle between the surfaces $x^2+y^2+z^2=2$ and $z=x^2+y^2-2$ at the point $(1,-1,0)$.
   (4) (3%) Let $\underline{\mathbf{F}}=\dfrac{y}{x^2+y^2}\underline{\mathbf{i}}+\dfrac{-x}{x^2+y^2}\underline{\mathbf{j}}$ be a 2-D vector field. Evaluate $\oint_C \underline{\mathbf{F}}\cdot d\underline{\mathbf{r}}$ over a closed contour C given by:



(5) (3%) Let $\underline{\mathbf{v}}(x,y,z)=\dfrac{x}{yz}\,\underline{\mathbf{i}}+\dfrac{y}{zx}\,\underline{\mathbf{j}}-\dfrac{z}{xy}\,\underline{\mathbf{k}}$. Evaluate the integral $\oint\!\!\int_S \underline{\mathbf{n}}\cdot(\nabla\times\underline{\mathbf{v}})\,dS$ over the surface of a sphere S given by: $(x-2)^2+(y-2)^2+(z-2)^2=1$. ($\underline{\mathbf{n}}$ denotes unit vector normal to the surface of the sphere S.)

7. Let $z=x+iy$ denote the complex variable, $\bar{z}=x-iy$ the complex conjugate of z, and $f(z)$ a complex function. Answer the following questions. **(Derivations are not required.)**

(1) (3%) $\oint_C dz/\bar{z}=2\pi i$ for any simple closed curve C enclosing $\bar{z}=0$. (True or False)

(2) (3%) Find the residue of the complex function $f(z)=\dfrac{e^{iz}}{z^4\cos z}$ at $z=0$.

(3) (3%) If the Laurent series expansion of $f(z)=z^2/(z^2+1)$ about $z=i$ is denoted by $\sum_{n=-\infty}^{n=+\infty}a_n(z-i)^n$, find $\left|\sum_{n=-\infty}^{n=+\infty}a_n\right|=?$

(4) (3%) Evaluate the complex integral $\oint_C \dfrac{e^{-i\bar{z}}}{\bar{z}}dz$ over C: $|z|=1$.

(5) (3%) Evaluate the real integral $\displaystyle\int_{\theta=0}^{\theta=2\pi}\frac{d\theta}{1+a^2-2a\cos\theta}$ with $0<a<1$.