※ 注意：不准使用任何計算機或電子儀器

Any device with computer algebra system is prohibited during the exam.

PART 1 : Fill in the blanks.

- Please ensure that each answer is clearly labeled with the corresponding blank number.
- Please note that only the final answers will be graded, and each blank is worth 5 points.

1. Suppose that $f(x)$ is differentiable at $x = 1$. Evaluate the following limit in terms of $f(1)$ and $f'(1)$.
$$\lim_{a\to 0} \frac{\sqrt{f(e^{2a})} - \sqrt{f(1)}}{\log_2(1-3a)} = \underline{\quad (1) \quad}.$$

2. Suppose that
$$\sqrt{3+y}\, x^3 - xy - 1 = 0.$$
At $(x, y) = (1, 1)$, $\frac{dx}{dy} = \underline{\quad (2) \quad}$. By the linear approximation, we can approximate the real root of $\sqrt{4.1}\, x^3 - 1.1x - 1 = 0$ with $\underline{\quad (3) \quad}$.

3. Consider the curve satisfying $5x^2 + 2xy + y^2 = 16$. The higest point of the curve (point with the largest y coordinate) is $\underline{\quad (4) \quad}$.

4. Suppose that $f(u) > 0$. Let
$$F(x) = \int_0^{x^2} \int_4^t f(u)\, du\, dt.$$
On what intervals is $F(x)$ increasing? $\underline{\quad (5) \quad}$

5. Let $f(x) = x^3 + 2^x + 1$ and $g(x) = f^{-1}(x)$, the inverse function of $f(x)$. Then $g'(4) = \underline{\quad (6) \quad}$ and $\int_2^4 g(x)\, dx = \underline{\quad (7) \quad}$.

6. Use the Maclaurin series of $\frac{\sin(x^2)}{x}$ to write the integral as the sum of an infinite series. $\int_0^{\frac{1}{2}} \frac{\sin(x^2)}{x}\, dx = \underline{\quad (8) \quad}$.

7. Assume that
$$|\, f(x, y) - 1 + 2y \,| \le \sin(x^2 + y^2) \quad \text{for } x^2 + y^2 \le 1\,.$$
Then the tangent plane of $y = f(x, y)$ at $(0, 0, f(0, 0))$ is $\underline{\quad (9) \quad}$. The tangent line of the level curve $f(x, y) = f(0, 0)$ at $(0, 0)$ is $\underline{\quad (10) \quad}$. The maximum value of directional derivatives of f at $(0, 0)$, $D_{\mathbf{u}} f(0, 0)$, is $\underline{\quad (11) \quad}$.

8. Find critical points of $f(x, y) = -2x^4 + x^2 y - y^2 + 7y$ and indicate whether they are local maximum, local minimum, or saddle points. $\underline{\quad (12) \quad}$.

9. a $\int_0^1 \int_{\sin^{-1} y}^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{1 + \cos^2 x}}\, dx dy = \underline{\quad (13) \quad}$.

   b $\int_0^2 \int_0^{\sqrt{2x - x^2}} \sqrt{x^2 + y^2}\, dy dx = \underline{\quad (14) \quad}$.

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PART 2 :

- Please solve the following problems and provide explanations and computations.
- Partial credits are allocated according to the level of completeness in your work.

1. $f(x) = \begin{cases} \frac{2e^{-x}}{(1+e^{-x})^2} & , \text{if } x \geq 0, \\ 0 & , \text{if } x < 0. \end{cases}$ is the probability density function of a random variable X.

   (a) (10 points) Sketch the graph of $f(x)$, indicating intervals of incerase/decrease, inflection point(s), and the horizontal asymptote.

   (b) (10 points) Evaluate the expected value of X which is $\int_0^\infty x f(x)\, dx$.

2. (10 points) The plane $x + 2y + z = 2$ intersects the cone $y = x^2 + z^2$ in an ellipse. Find the points on the ellipse that are nearest and farthest from the origin.

試題隨卷繳回