※ Please show the detailed calculation process for the questions whenever necessary.
※ If the "final" answers are with decimal numbers, please round to the fourth decimal place, e.g., 99.3745 or 0.0243 = 2.43%.

1. (25%) If z follows a normal distribution, i.e., $z \sim ND(\mu_z, \sigma_z^2)$, its probability density and cumulative distribution functions are

$$n_1(z; \mu_z, \sigma_z^2) = \frac{1}{\sqrt{2\pi}\sigma_z} \exp\left(-\frac{1}{2}\left(\frac{x-\mu_z}{\sigma_z}\right)^2\right),$$

and

$$N_1(z^*; \mu_z, \sigma_z^2) = \int_{-\infty}^{z^*} n_1(z; \mu_z, \sigma_z^2)\, dx,$$

respectively. If $\begin{bmatrix} x \\ y \end{bmatrix}$ follows a bivariate normal distribution, i.e.,

$$\begin{bmatrix} x \\ y \end{bmatrix} \sim ND\left(\begin{bmatrix} \mu_x \\ \mu_y \end{bmatrix}, \begin{bmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{bmatrix}\right),$$

the corresponding probability density and cumulative distribution functions are

$$n_2\left(\begin{bmatrix} x \\ y \end{bmatrix}; \mu_x, \mu_y, \sigma_x^2, \sigma_y^2, \rho\right) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \cdot \exp\left(-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right) + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]\right),$$

and

$$N_2\left(\begin{bmatrix} x^* \\ y^* \end{bmatrix}; \mu_x, \mu_y, \sigma_x^2, \sigma_y^2, \rho\right) = \int_{-\infty}^{x^*}\int_{-\infty}^{y^*} n_2\left(\begin{bmatrix} x \\ y \end{bmatrix}; \mu_x, \mu_y, \sigma_x^2, \sigma_y^2, \rho\right) dydx,$$

respectively.

(a) (10%) Given $\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \sim ND\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$, express $x = \alpha z_1 + \beta$ and $y = \gamma z_1 + \delta z_2 + \varepsilon$ such that

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$$\begin{bmatrix} x \\ y \end{bmatrix} \sim ND\left(\begin{bmatrix} \mu_x \\ \mu_y \end{bmatrix}, \begin{bmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{bmatrix}\right).$$

What are α, β, γ, δ, and ε?

(b) (15%) Evaluate the expectation $E[\exp(x+y)I\{x \le 0\}]$, where $I\{A\}$ is an indicator function returning 1 (0) if the condition A is true (false), and express the calculation result as $a + bn_1(c;0,1) + dN_1(e;0,1) + fn_1(g;0,1)n_1(h;0,1) + in_1(j;0,1)N_1(k;0,1) + lN_2\left(\begin{bmatrix} m \\ n \end{bmatrix};0,0,1,1,0\right)$. What are a, b, c, d, e, f, g, h, i, j, k, l, m, and n?

2. (15%) One of the debt models proposed by Evsey Domar, a Keynesian economist, can be described as follows. Let D and I respectively denote the national debt and national income, and assume that both are functions of time t. Furthermore, both the time rates of change of D and I are assumed to be proportional to I, so that

$$\frac{dD}{dt} = \alpha I \text{ and } \frac{dI}{dt} = rI.$$

Suppose $I(0) = I_0$ and $D(0) = D_0$.

(a) (10%) Solve both of these differential equations and express $D(t)$ and $I(t)$ in terms of α, r, I_0, and D_0.

(b) (5%) Determine what happens to this ratio in the long run by computing the limit

$$\lim_{t\to\infty} \frac{D(t)}{I(t)}$$

3. (10%) IB company produces x units of commodity A and y units of commodity B. All the units can be sold for $p = 20 - 5x$ dollars per unit of A and $q = 4 - 2y$ dollars per unit of B. The cost (in dollars) of producing these units is given by the joint-cost function $C(x,y) = 2xy + 4$. What should x and y be to maximize profit?

4. (10%) Find $\frac{dy}{dx}$ for the equation $y^3 + y^2 - 5y - x^2 = -4$.

5. (10%) Evaluate $\int \frac{\sin\sqrt{x}}{\sqrt{x}} dx$.

6. (10%) Find the value of $\int_{\frac{\pi}{8}}^{\frac{\pi}{6}} (\csc 4x - \cot 4x) dx$.

7. (10%) Use a Maclaurin polynomial to find the value of $\sqrt{e}$ accurate to four decimal places.

8. (10%) Given $f(x,y) = e^x \sin y + \ln xy$, find $\frac{\partial^3 f}{\partial x \partial y^2}$.

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