1. (20 pts) Let $a, b \in \mathbb{R}$.
(a) Find all (a, b) such that $x^a \sin(x^b)$ is uniformly continuous on $(0, 1]$.
(b) Find all (a, b) such that $e^{ax} \sin(e^{bx})$ is uniformly continuous on $[0, \infty)$.

2. (20 pts) Assume $a > 0$.
(a) Let $f_n(x) = \dfrac{n^a x}{nx^2 + 1}$. Find all a such that f_n is uniformly convergent on $[0, 1]$ as $n \to \infty$.
(b) Let $f_n(x) = n^a x^n (1 - x)$. Find all a such that f_n is uniformly convergent on $[0, 1]$ as $n \to \infty$.

3. (20 pts) Let

$$\alpha = \frac{1}{2^2} + \frac{1}{6^2} + \frac{1}{10^2} + \frac{1}{14^2} + \frac{1}{18^2} + \frac{1}{22^2} + \cdots.$$

(a) Prove that the above series is convergent.
(b) Find the value of α by the method of the Fouries series.

4. (20 pts) Assume $\lim_{n\to\infty} a_n = 5$ and $\lim_{n\to\infty} b_n = 3$. Let

$$\alpha = \lim_{n\to\infty} \frac{a_1 b_1 + a_2 b_2 + \cdots + a_n b_n}{a_1 + a_2 + \cdots + a_n}$$

and

$$\beta = \lim_{n\to\infty} \frac{\sum_{k=1}^{n-1} k a_k b_{n-k}}{n^2}.$$

Find α and β. Verify your answer.

5. (20 pts) Let f and g be bounded functions. Prove that $f + g$ is Riemann integrable on $[0, 1]$ if both f and g are Riemann integrable on $[0, 1]$.