本卷滿分為 100 分，總得分為負時以 0 分計。

1. 是非題

(每題答對/答錯/不作答分別得 2/-1/0 分。答案卷上請標明題號並依序以 ○/× 分別表示「是/非」作答)

1. A subset S of a metric space is compact if it is bounded and closed.
2. If f is a C^∞ function on $\mathbf{R}$ and its Taylor series with center 0 converges everywhere on $\mathbf{R}$, then the series converges to f pointwise at least in some open neighborhood of 0.
3. A convex function on $[-1,1]$ must be continuous.
4. A convex function on $\mathbf{R}$ must be continuous.
5. If a function $\mathbf{R} \xrightarrow{f} \mathbf{R}$ has the intermediate value property (i.e., for any $a<b$ and any number c between $f(a)$ and $f(b)$ there exists some $x_0 \in [a,b]$ such that $c = f(x_0)$), then f is continuous.
6. If f_n is a sequence of Riemann integrable functions on $[0,1]$ which converges pointwise to a a function f, then f has at most countably many discontinuous points.
7. If a map $[0,1] \xrightarrow{f} \mathbf{R}$ is uniform continuous, then it is Lipschitz.
8. Consider the map $(x,y,z) \in \mathbf{R}^3 \xmapsto{f} (x^3, x^2+y, z^5 + z\sin^2(xy)) \in \mathbf{R}^3$. For every open set U in $\mathbf{R}^3$ containing $(0,0,0)$ the restriction $f|_U$ is not an injection.
9. Suppose that $[0,1] \xrightarrow{f} \mathbf{R}$ is Riemann integrable on $[c,1]$ for every $c \in (0,1]$. If $\lim\limits_{c\to 0^+} \int_c^1 f(x)\,dx$ exists, then f is also Riemann integrable on $[0,1]$ and $\int_0^1 f(x)\,dx = \lim\limits_{c\to 0^+} \int_c^1 f(x)\,dx$.
10. If $f(x,y)$ is a C^2 function and $\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0$, $\frac{\partial^2 f}{\partial x^2}(0,0) < 0$, and $\frac{\partial^2 f}{\partial y^2}(0,0) < 0$, then f has a local maximum at $(0,0)$.

2. 填充、計算與證明

(請在答案卷上標明題號，作答時不需依照題目編號順序。注意時間，先做有把握的題目。)

1.(10 points) Let $A = \{(x,y,z,w) \in \mathbf{R}^4 \,|\, x^2+y^2+z^2-w^3 \le 1, x^2+y^2+z^2 \le 4, 0 \le w \le 10\}$. Fill in the boxes in the expression below with functions so that both equalities hold for <u>all</u> continuous functions f on $\mathbf{R}^4$:

$$\int_A f = \int_{\square}^{10}\left(\int_{\square}^{\square}\left(\int_{\square}^{\square}\left(\int_{\square}^{\square} f(x,y,z,w)dx\right)dy\right)dz\right)dw = \int_{\square}^{\square}\left(\int_{\square}^{\square}\left(\int_{\square}^{\square}\left(\int_{\square}^{\square} f(x,y,z,w)dw\right)dy\right)dx\right)dz.$$

2.(15 points) Consider the vector field $(P(x,y), Q(x,y)) := \left(y + e^{xy}, \frac{xy-1}{y^2}e^{xy} + 3x\right)$ on $\mathbf{R}^2 \setminus (\mathbf{R} \times \{0\})$. Compute $\int_\gamma P dx + Q dy$ where $\gamma(t) = (\cos t, \sin t)$, $t \in [\frac{\pi}{6}, \frac{5\pi}{6}]$.

3.(15 points) Let X be a metric space. We say that a function $X \xrightarrow{f} \mathbf{R}$ is *upper semicontinuous* (usc for short) if $\limsup\limits_{x\to a} f(x) \leqslant f(a)$ for every $a \in X$. Show that if $X \xrightarrow{f} \mathbf{R}$ is usc and X is compact, then f achieves its maximum on X, i.e., there exists $a \in X$ such that $f(a) \geqslant f(x)$ for all $x \in X$.

4.(15 points) Let $X \xrightarrow{f} X$ be a self map on a complete metric space (X,d). Show that if there exists a constant $0 \leqslant C < 1$ such that $d(f(x), f(x')) \leqslant C d(x,x')$ for all $x, x' \in X$ then there exists a unique $x_0 \in X$ such that $f(x_0) = x_0$.

5. Let $\mathbf{R} \xrightarrow{f} \mathbf{R}$ be a function such that $\lim\limits_{t\to a} f(t)$ exists for <u>every</u> $a \in \mathbf{R}$.

(1)(10 points) Show that the function $x \in \mathbf{R} \xmapsto{g} \lim\limits_{t\to x} f(t) \in \mathbf{R}$ is continuous.

(2)(15 points) Let g be the function defined in (1). Show that for every $\epsilon > 0$ and every pair of real numbers a, b with $a < b$ the set $A := \{x \in [a,b] \,|\, |f(x) - g(x)| > \epsilon\}$ is finite. (Hint. You might want to use the Bolzano-Weierstrass theorem.)

試題隨卷繳回