※ 注意：請於試卷內之「非選擇題作答區」作答，並應註明作答之題號。

In the whole test, we use the following notations:

$C[-1,1]=\{g:[-1,1]\to\mathbb{R} \mid g \text{ is continuous on } [-1,1]\}$ is a norm linear space

with the sup-norm $\|g\|_\infty=\sup_{x\in[-1,1]}|g(x)|$.

1. 20%

Prove that $C[-1,1]$ is complete.

2. 30% (each 10%)

Let $A=\{f\in C[-1,1] \mid \|f\|_\infty\le 1\}$,

$$B=\left\{f\in A \mid \sup_{x\neq y}\frac{|f(x)-f(y)|}{|x-y|}\le 1\right\},$$

$$C=\{f\in C[-1,1] \mid \|f\|_\infty>1\}.$$

(i) Which one is sequentially compact in $C[-1,1]$?

(ii) Which one is open in $C[-1,1]$?

(iii) Which one is NOT convex ?
Justify all your answers.

3. 20%

Find a sequence of nonnegative and integrable functions $\{f_k\}_{k=1}^{\infty}$ such that each $f_k:\mathbb{R}\to\mathbb{R}$ is continuous, $\int_{\mathbb{R}} f_k(x)\,dx=1$, for $k=1,2,3,\ldots$, and $\lim_{k\to\infty} f_k(0)=\infty$, $\lim_{k\to\infty} f_k(x)=0, \forall x\neq 0$. Calculate $\lim_{k\to\infty}\int_{\mathbb{R}}(x+1)^{20} f_k(x)\,dx=?$ Justify your answer.

見背面

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4. 30% (each 10%)

Let $f:\mathbb{R}^n \to \mathbb{R}$ be continuous and $0<\int_{\mathbb{R}^n}|f(x)|dx<\infty$, where $n\geq 2$.

(i) Calculate

$\lim_{\varepsilon\to 0+}\varepsilon^{-1}\int_{B_\varepsilon} f(x)|x|^{1-n}\,dx$, where $n\geq 2, B_\varepsilon=\{x\in\mathbb{R}^n \mid |x|<\varepsilon\}$,

and $|x|=\sqrt{\sum_{j=1}^{n}x_j^2}$ for $x=(x_1,\cdots,x_n)\in\mathbb{R}^n$.

(ii) Suppose that function f is radially symmetric i.e. $f(x)=f(|x|)$ for $x\in\mathbb{R}^n$. Let $\tilde{f}(\xi)=\int_{\mathbb{R}^n} f(x)\cos(x\cdot\xi)dx$ for $\xi\in\mathbb{R}^n$. Prove that $\tilde{f}$ is radially symmetric i.e. $\tilde{f}(\xi)=\tilde{f}(|\xi|)$ for $\xi\in\mathbb{R}^n$. Note that f is radially symmetric $\Leftrightarrow$ $f(\mathrm{M}x)=f(x)$ for any $x\in\mathbb{R}^n$, and any orthogonal matrix $\mathrm{M}\in\mathbb{R}^{n\times n}$ $(\mathrm{M}^T\mathrm{M}=\mathrm{M}\mathrm{M}^T=I)$.

(iii) Calculate $\lim_{|\xi|\to\infty}\tilde{f}(\xi)=?$

Justify all your answers.

試題隨卷繳回