Notation: $\mathbf{R}$ is the set of real numbers, and $\mathbf{C}$ is the set of complex numbers. If $F = \mathbf{R}$ or $\mathbf{C}$, denote by $M_n(F)$ the $n \times n$ matrices with entries in F. If $A \in M_{m\times n}(F)$, denote by $A^t \in M_{n\times m}(F)$ the transpose of A. Denote by I_n the $n \times n$ identity matrix and 0_n the $n \times n$ zero matrix.

Problem 1 (10pts). Let $\mathbf{i} = \sqrt{-1} \in \mathbf{C}$ be a root of $X^2 + 1$. Let

$$v_1 = (1, 0, -\mathbf{i}), \quad v_2 = (1 + \mathbf{i}, 1 - \mathbf{i}, 1), \quad v_3 = (\mathbf{i}, \mathbf{i}, \mathbf{i}).$$

Show that $\{v_1, v_2, v_3\}$ is a basis of $\mathbf{C}^3$ and express the vector $v_4 = (1, 0, 1)$ as a linear combination of v_1, v_2 and v_3, namely find $\alpha_1, \alpha_2, \alpha_3 \in \mathbf{C}$ such that $v_4 = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$.

Problem 2 (15 pts). Let

$$\mathbf{v}_1 = (0, 3, 3, 1),\ \mathbf{v}_2 = (2, 1, -3, 7),\ \mathbf{v}_3 = (1, 8, 6, 6),\ \mathbf{v}_4 = (1, 10, -4, 2)$$

be vectors in $\mathbf{R}^4$. Let $W_1 = \mathrm{span}_{\mathbf{R}}\{\mathbf{v}_1, \mathbf{v}_2\}$ and let $W_2 = \mathrm{span}_{\mathbf{R}}\{\mathbf{v}_3, \mathbf{v}_4\}$. Find the dimension and a basis of $W_1 \cap W_2$.

Problem 3 (25 pts). Let

$$A = \begin{pmatrix} 1 & 0 & 1 \\ -2 & 1 & 0 \end{pmatrix} \in M_{2\times 3}(\mathbf{R}).$$

(1) (15pts) Find an orthogonal matrix $P \in M_3(\mathbf{R})$ such that $P^{-1}A^tAP$ is a diagonal matrix.

(2) (10pts) Find the singular value decomposition of A. In other words, factorize $A = U\Sigma V^t$, where $U \in M_3(\mathbf{R})$ and $V \in M_3(\mathbf{R})$ are orthogonal matrices and $\Sigma \in M_{2\times 3}(\mathbf{R})$ is of the form

$$\Sigma = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \end{pmatrix}, \quad \lambda_1 \geq \lambda_2 \geq 0$$

Problem 4 (15pts). Let $V = M_3(\mathbf{C})$ be a 9-dimension vector space over $\mathbf{C}$ and let

$$A = \begin{pmatrix} 0 & 0 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{pmatrix}.$$

Define the linear transformation $T : V \to V$ by

$$T(B) = AB - BA.$$

(1) (5pts) Find the dimension of $\mathrm{Ker}\, T$.

(2) (10pts) Show that T is diagonalizable.

Problem 5 (15pts). Let $A, B \in M_n(\mathbf{R})$. Prove that $\mathrm{rank}\, A + \mathrm{rank}\, B \leq n$ if and only if there exists an invertible matrix $X \in M_n(\mathbf{R})$ such that $AXB = 0_n$.

Problem 6 (20pts). Let A and B be elements in $M_n(\mathbf{C})$. Suppose that

$$AB - BA = c \cdot (A - B)$$

for some non-zero $c \in \mathbf{C}$. Prove that there exists an invertible matrix $P \in M_n(\mathbf{C})$ such that $P^{-1}AP$ and $P^{-1}BP$ are upper-triangular matrices with the same diagonal entries.