國立臺灣大學 114 學年度碩士班招生考試試題

1. Let $\mathbf{A}$ be an $n \times n$ matrix. Write down the definitions of the following terms:
   - (a) $\mathbf{A}$ is idempotent. (2 points)
   - (b) $\mathbf{A}$ is orthogonal. (2 points)
   - (c) $\mathbf{A}$ is positive-definite. (2 points)
   - (d) The null space of $\mathbf{A}$. (2 points)
   - (e) The rank of $\mathbf{A}$. (2 points)

2. Let $\mathbf{A}$ be an $m \times n$ matrix. Consider the Frobenius norm of $\mathbf{A}$, defined as $\|\mathbf{A}\|^2 = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2 = \text{tr}(\mathbf{A}^T\mathbf{A})$. Show that $\|\mathbf{PAQ}\| = \|\mathbf{A}\|$ for any $m \times m$ orthogonal matrix $\mathbf{P}$ and any $n \times n$ orthogonal matrix $\mathbf{Q}$. (10 points)

3. (a) Compute $\lim_{x\to 0} \frac{\sqrt{4+x}-2}{|x^2-x|}$ or show that it does not exist. (5 points)
   - (b) Compute $\frac{d}{dx}\left(2^{2^x} + x^{x^2}\right)$. (5 points)
   - (c) Compute $\int_0^\infty e^{-2x}\sin(3x)dx$. (5 points)
   - (d) Compute $\int \log(x^2+1)\,dx$. (5 points)

4. Let $\mathbf{X}$ be an $n \times p$ matrix with $n > p$. Consider the matrix $\mathbf{X}^T\mathbf{X} + \xi\mathbf{I}$ where $\xi > 0$. Show that the smallest eigenvalue of $\mathbf{X}^T\mathbf{X} + \xi\mathbf{I}$ is greater than or equal to ξ. (10 points)

5. Consider the following quadratic form

$$f(x, y, z) = x^2 - 2y^2 + 2xy + 2xz - 2yz$$

   - (a) Write down the symmetric matrix $\mathbf{A}$ associated with the quadratic form, i.e., find $\mathbf{A}$ such that

$$f(x,y,z) = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \text{ where } \mathbf{A} = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}.$$

(3 points)、

   - (b) Determine whether $(0,0,0)$ is a local maximum/minimum or saddle point for f. (7 points)

6. Maximize $f(x,y,z) = yz$ subject to $x + z = 1$ and $x^2 + y^2 \leq 6$.
   - (a) Write out the Lagrangian function and the first order conditions. (10 points)
   - (b) Solve the constrained optimization problem. (10 points)

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7. Let W be a subspace of an inner product space V and $b_1, b_2, \ldots, b_k$ be an orthonormal basis for W. For each vector $v \in V$, let $P(v)$ be its projection onto W.

   (a) Show that the range of $I - P$ is the orthogonal complement of W and the kernel of $I - P$ is W. (10 points)

   (b) If $v \in V$, show that $P(v)$ is the vector in W closest to v. (10 points)

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