# Statistics

1. Let $X_1, \ldots, X_n$ be independent Bernoulli random variables with $p_i = \mathbb{P}(X_i = 1)$, $i = 1, \ldots, n$, such that

$$\log \frac{p_i}{1-p_i} = \alpha + \beta t_i,$$

where $t_i$'s are known constants while $\alpha$ and $\beta$ are unknown parameters. Please

   (a) show that the pmf of $X_i$ can be expressed by

$$\mathbb{P}(X_i = x_i) = \frac{\exp(x_i(\alpha + \beta t_i))}{1 + \exp(\alpha + \beta t_i)}, \quad x_i = 0 \text{ or } 1,$$

   and derive the joint pmf of $X_1, \ldots, X_n$, (5 points)

   (b) show that the joint pmf forms an exponential family and find a 2-dimensional sufficient statistic for $(\alpha, \beta)$. (5 points)

2. The pdf of $\Gamma(\alpha, \lambda)$ is given as

$$f(x|\alpha, \lambda) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \exp(-\lambda x), \quad x > 0; \ \alpha > 0, \ \lambda > 0.$$

Let $X_1, X_2, \ldots, X_n \overset{\text{i.i.d.}}{\sim} \Gamma(\alpha, \lambda)$. Please

   (a) find the method of moment estimators $(\hat{\alpha}, \hat{\lambda})$ for $(\alpha, \lambda)$, (10 points)

   (b) show that both $\hat{\alpha}$ and $\hat{\lambda}$ are consistent. (10 points)

3. Let $X_1$ and $X_2$ be two i.i.d. random variables from a location-scale family with pdf

$$f_{\theta,\sigma}(x) = \frac{1}{\sigma} g\left(\frac{x-\theta}{\sigma}\right), \quad x \in \mathbb{R},$$

where $\sigma > 0$, $\theta \in \mathbb{R}$, and $g$ is a known pdf that does not depend on $\theta$ and $\sigma$. Please

   (a) given

$$Z = \frac{X_1 + X_2 - 2\theta}{|X_1 - X_2|},$$

   show that the pdf of $Z$ does not depend on the parameters $\theta$ and $\sigma$, (5 points)

   (b) derive a $(1-\alpha)$ confidence interval for $\theta$ using $q_{\alpha/2}$ and $q_{1-\alpha/2}$, the upper and lower $(\alpha/2)$th quantiles for the distribution of $Z$ respectively. (5 points)

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4. Consider the following hierarchical model:

$$Z \mid p \sim \text{Ber}(p), \quad 0 < p < 1,$$
$$X \mid Z = 0, \theta_1 \sim \text{Poisson}(\theta_1), \quad \theta_1 > 0,$$
$$X \mid Z = 1, \theta_2 \sim \text{Geom}\left(\frac{1}{\theta_2 + 1}\right), \quad \theta_2 > 0.$$

Please

(a) derive the joint likelihood function for $(X, Z)$, (5 points)

(b) given $\{(x_i, z_i),\ i = 1, \ldots, n\}$ i.i.d. from the above model, find the MLEs of $\theta_1$, $\theta_2$, and $p$. (10 points)

(Recall the pmf of Geom$(p)$ is given by $\mathbb{P}(X = x) = p(1-p)^x$ for $x = 0, 1, 2, \ldots$ and the pmf of Poisso$(\lambda)$, $\lambda > 0$, is given by $\mathbb{P}(X = x) = (e^{-\lambda}\lambda^x)/x!$ for $x = 0, 1, 2, \ldots$.)

5. Let $X_1, \ldots, X_n$ be i.i.d. Bernoulli random variables with success probability $p$. Let $\hat{p} = \sum_{i=1}^n X_i/n$. Please

(a) show that $\sqrt{n}\left(\hat{p}^2 - p^2\right)$ converges in distribution to $N\left(0, 4p^3(1-p)\right)$, (5 points)

(b) show that $4X_1X_2X_3(1 - X_4)$ is an unbiased estimator of $4p^3(1-p)$, (2 points)

(c) find the UMVUE of $4p^3(1-p)$ when $n \geq 4$. (8 points)

6. Suppose $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ are i.i.d. from $p$-dimensional normal $N_p(\boldsymbol{\mu}, \boldsymbol{I}_p)$, where $\boldsymbol{\mu} \in \mathbb{R}^p$ and $\boldsymbol{I}_p$ is a $p$-dimensional identity matrix, whose pdf is given by

$$f(\boldsymbol{x}|\boldsymbol{\mu}) = \frac{1}{(2\pi)^{p/2}} \exp\left\{-\frac{1}{2}(\boldsymbol{x} - \boldsymbol{\mu})^\top (\boldsymbol{x} - \boldsymbol{\mu})\right\}, \quad \boldsymbol{x} = (x_1, \ldots, x_p)^\top \in \mathbb{R}^p.$$

Please

(a) show that the log-likelihood ratio test statistic to test $H_0 : \|\boldsymbol{\mu}\| = r$ versus $H_1 : \|\boldsymbol{\mu}\| \neq r$ is equivalent to $T = n(\|\bar{\boldsymbol{X}}\| - r)^2$, where $\|\boldsymbol{\mu}\|^2 = \sum_{i=1}^p \mu_i^2$ for $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_p)^\top$ and $r$ is a fixed constant, (10 points)

(b) assuming $nr^2 > c$ for some constant $c$, please derive the power of the likelihood ratio test that rejects $H_0$ when $T > c$ in terms of the cdf for a noncentral chi-squared distribution. (10 points)
(Recall that given $k$ independent normally distributed random variables $X_i \sim N(\mu_i, 1)$, $i = 1, \ldots, k$, $\sum_{i=1}^k X_i^2$ follows the noncentral chi-squared distribution with degrees of freedom $k$ and noncentrality $\lambda = \sum_{i=1}^k \mu_i^2$.)

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7. Consider a Bayesian model where the random parameter $\Theta$ has a Bernoulli prior distribution such that $\mathbb{P}(\Theta = 0) = \pi$ and $\mathbb{P}(\Theta = 1) = 1 - \pi$ with $0 < \pi < 1$. Given $\Theta = 0$, the data $X$ has pdf $f_0(x)$, and given $\Theta = 1$, $X$ has pdf $f_1(x)$. With a loss function $L(\theta, d)$, the Bayes estimator $\delta^{\pi}(X)$ minimizes the posterior expected loss $\mathbb{E}\left[L(\Theta, \delta^{\pi}(X))\,|X = x\right]$. Please find the Bayes estimator of $\Theta$ under

   (a) squared error loss $L(\theta, d) = (\theta - d)^2$, (5 points)

   (b) $0 - 1$ loss, (5 points)

$$L(\theta, d) = \begin{cases} 0, & d = \theta, \\ 1, & \text{o.w..} \end{cases}$$

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