1. (25 pts) Solve the following equations.
   (a) $\frac{dy}{dx} = 1 + (2 - x + y)^3$.
   (b) $e^x y \frac{dy}{dx} + (1 + y^2) = 0$.
   (c) $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0, y(0) = 1, y'(0) = 0$.

2. (30 pts)
   Let $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ -4 & 5 \end{pmatrix}$.
   (a) Find $e^{\mathbf{A}t}$.
   (b) Solve the system $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$, $\mathbf{x}(0) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$.
   (c) Solve the system $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\mathbf{x}(0) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$.

3. (25 pts) Assume that both $y_1(x)$ and $y_2(x)$ have continuous second derivatives on $[0, \pi]$ and satisfies

$$\text{(E1)} \quad \begin{cases} y'' + y = 0 \text{ for } x \in [0, \pi/2] \\ y' + xy = 0 \text{ for } x \in [\pi/2, \pi]. \end{cases}$$

   (a) Find the solution $y_1(x)$ with $y_1(\pi) = 1$.
   (b) Is it possible that $y_2(0) = 1$?
   (c) Let $S = \{y : [0, \pi] \to \mathbb{R} \mid y \text{ has a continuous second derivative and satisfies (E1)}\}$. Show that S is a vector space over $\mathbb{R}$ and find its dimension.

4. (20 pts) Let $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a real matrix with $b \neq 0$ and $c \neq 0$. Assume that

$$\text{(E2)} \quad \mathbf{y}'(t) = \mathbf{A}\mathbf{x}(t), \quad \text{where } \mathbf{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(x) \end{pmatrix}, 0 \leq t \leq 1.$$

   (a) Show that there is a polynomial $P(x)$ of degree 2 such that $P(\mathbf{A}) = \mathbf{0}_{2\times 2}$.
   (b) Show that there is a polynomial $Q_1(x)$ of degree 2 such that $Q(y_1(t)) = 0$.
   (c) Is it true that $Q_2(x) = \beta P(x)$ for some constant β if $Q_2(x)$ is a polynomial of degree 2 and $Q_2(y_1(t)) = 0$ whenever (E2) holds?

試題隨卷繳回