Notations:
$\mathbb{Q}$: the field of rational numbers.
$\mathbb{F}_n$: the finite field with n elements.
$Z(G)$: the center of the group G.

1. (15 %) Let R be a commutative ring with 1 and $I_1, \ldots, I_n$ be ideals in R. Show that if I_i, I_j are coprime $\forall i \neq j$, then

$$R/I_1 I_2 \cdots I_n \cong R/I_1 \times R/I_2 \times \cdots \times R/I_n.$$

2. (15%) Let A_d be the ring of integers in the quadratic field $\mathbb{Q}(\sqrt{d})$.
   (a) Show that A_5 is a UFD.
   (b) Show that A_{-5} is not a UFD.

3. (30%)
   (a) Show that if $|G| = 3 \times 5 \times 7^2 \times 13$ and N is a normal subgroup of G with $|N| = 5$, then $N \subset Z(G)$.
   (b) Prove that no simple group has order $p^2 q$, where p and q are primes.
   (c) Show that if $|G| = 225$, then there is a normal subgroup having prime index.

4. (24 %)
   (a) Let L/K be a finite separable extension. Show that L has a primitive element α over K, that is, $L = K(\alpha)$.
   (b) Let x and y be indeterminates and let $L = \mathbb{F}_p(x, y)$, $K = \mathbb{F}_p(x^p, y^p)$ with p a prime. Show that L does not have a primitive element over K

5. (16 %)
   (a) Let K be a field and $f(x) \in K[x]$ be a separable polynomial. Assume that $f(x) = g(x)h(x)$ in $K[x]$. Find an example of $f(x)$ such that the Galois groups of $f(x)$, $g(x)$ and $h(x)$ over K are nontrivial and the Galois group of $f(x)$ over K is isomorphic to the direct product of the Galois groups of $g(x)$ and $h(x)$ over K. (Justify your answer)
   (b) Prove that the Galois group of $(x^3-2)(x^3-3)$ over $\mathbb{Q}$ is not isomorphic to the direct product of the Galois groups of $x^3 - 2$ and $x^3 - 3$ over $\mathbb{Q}$.