※ Please write down the detailed calculation process for all questions.

1. (20%) The Black-Scholes formula for a call option with six input parameters (S, X, r, q, σ, T) is as follows.
$$c(S, X, r, q, \sigma, T) = Se^{-qT}N(d_1) - Xe^{-rT}N(d_2),$$
where
$$d_1 = \frac{\ln(S/X)+(r-q+\sigma^2/2)T}{\sigma\sqrt{T}} \text{ and } d_2 = \frac{\ln(S/X)+(r-q-\sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T},$$
and $N(\cdot)$ is the cumulative distribution function of the standard normal distribution defined as
$$N(d) = \int_{-\infty}^{d} n(x)dx = \int_{-\infty}^{d} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx,$$
where $n(\cdot)$ is the probability density function of the standard normal distribution.

(a) (10%) Derive and express $\frac{\partial c}{\partial X}$ as the form of $AN(B)$. What are A and B?

(b) (10%) Derive and express $\frac{\partial c}{\partial T}$ as the form of $Cn(D) + EN(D) + FN(G)$. What are C, D, E, F, and G?

2. (10%) Consider that X follows a zero-mean normal distribution, i.e., $X \sim ND(0, \sigma^2)$. Evaluate the expectation $E[Xe^{cX}]$, where c is a constant real number. (Hint: The probability density function for $ND(\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$.)

3. (20%) Consider $J = \sum_{k=1}^{N} Y$ to be a compound Poisson distribution, where N follows a Poisson distribution with the expected number of event occurrences to be λ, and for each event occurrence, one receives a payoff Y, which follows an independent and identical normal distribution, i.e., $Y \sim ND(\mu, \sigma^2)$. Furthermore, assume N and Y are independent.
(a) (10%) Evaluate $E[\prod_{k=1}^{N} Y]$.
(b) (10%) Evaluate $E[N \prod_{k=1}^{N} Y]$.

(Hint: For N, the probability of observing k events is given by $P(k \text{ events occurring}) = \frac{e^{-\lambda}\lambda^k}{k!}$.)

4. (10%) Evaluate the integral $\int \frac{\sin x}{1+\sin x} dx$.

5. (10%) Perform the integration $\int \frac{dx}{\sqrt{x}-2}$.

6. (10%) Find *dy/dx* for the equation $e^{xy} - xy = 0$.

7. (10%) Find the derivative of *y* with respect to *x* in the function $y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$.

8. (10%) Solve the differential equation $(x^2 + 1)dx + \cos y\, dy = 0$.

試題隨卷繳回