1. **(20%)** For a maximization problem, the two-phase simplex method is used because a convenient initial solution is not available. In the tableau, there are five unknown constants **d**, **e**, **f**, **g**, and **h**. Assume that a_5 and a_6 are the artificial variables for phase 1. Give conditions or restrictions on these constants, so that the statements in the following questions are valid.

z	x_1	x_2	x_3	x_4	a_5	a_6	RHS
1	0	0	**d**	**e**	1	1	**f**
0	0	1	1	-1	2	1	**g**
0	1	0	1	**h**	1	1	20

(a) (4%) The current tableau is optimal in phase 1 and the original problem is feasible.

(b) (4%) Continue with (a), for the initial tableau in phase 2, what is the value of **d** if the objective function is "max $c_1x_1 + c_2x_2 + c_3x_3$"?

(c) (4%) For phase 2, the current solution is feasible but the solution is unbounded.

(d) (4%) For phase 2, the current solution is a degenerate basic feasible solution.

(e) (4%) For phase 2, the current tableau is optimal and an alternative solution exists.

2. **(20%)** Consider a sequencing problem on a single machine in which n jobs need to be processed by the machine. A job (say job j) is given its arrival time a_j, processing time p_j, and due date d_j. The machine cannot process job j until a_j and takes p_j (units of time) to complete the operation of job j. The completion time of job j is denoted as c_j and thus if c_j is larger than d_j, job j is late. The lateness of job j is defined as $L_j = \max(0, c_j - d_j)$, and the objective of the problem is to find the processing order of these jobs which minimizes $\sum_{j=1}^{n} L_j$. Let s_j (decision variable) denote the time the machine starts to process job j. The problem can be formulated as a mixed integer programming model and answer the following questions about the constraints of the model.

(a) (4%) List the constraints related to the arrival time of a job.

(b) (4%) Give the constraints about s_j and c_j.

(c) (6%) List the constraints about the lateness L_j.

(d) (6%) Let y_{ij} be decision variables of determining the sequence of jobs. If job i is processed before job j, y_{ij} =1; otherwise, $y_{ij} = 0$. Give the constraints that determine the processing order of jobs i and j.

3. **(10%)** (2.5% for each question) For an assignment problem, Hungarian algorithm is used to solve the problem. During the solving process, (a)_______ feasible is always maintained and (b)_______ feasible is tried to be achieved so that the optimal solution can be obtained. For a transportation problem, once an initial solution is generated in the first phase, an improved procedure in the second phase can be applied to find an optimal solution. Each iteration in the second phase, the transportation tableau always maintains (c)_______ feasible and tries to achieve (d)_______ feasible.

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4. (30%) Let f: $S \to \mathbf{R}$, where S is a nonempty convex set in $\mathbf{R}^n$. Then function f is said to be *convex* on S if

$$f\left(\lambda x_1+(1-\lambda)x_2\right)\le \lambda f(x_1)+(1-\lambda)f(x_2)$$

for each $x_1, x_2 \in S$ and for $\lambda \in [0,1]$. Then $\xi \in \mathbf{R}^n$ is a subgradient of f at $\bar{x}$ if $f(x)\ge f(\bar{x})+\xi^t(x-\bar{x})$ for all $x \in \mathbf{R}^n$. A set S in $\mathbf{R}^n$ is said to be *convex* if the line segment joining any two points of the set also belongs to the set. If x_1 and x_2 are in S, then $\lambda x_1+(1-\lambda)x_2$ must also belong to S for each $\lambda \in [0,1]$. We let $f:\mathbf{R}^n \to \mathbf{R}$ be a convex function. Let $\partial f(\bar{x})$ be the set of the subgradients of f at $\bar{x}$, i.e. $\partial f(\bar{x})=\left\{\xi \in \mathbf{R}^n : f(x)\ge f(\bar{x})+\xi^t(x-\bar{x})\ \ \forall x \in \mathbf{R}^n\right\}$.

(a) (5%) **(True or False)** Let $\xi_1,\xi_2 \in \partial f(\bar{x})$, $f(x)\ge f(\bar{x})+\xi_1^t(x-\bar{x})\ \ \forall x\in \mathbf{R}^n$ (Eq. 1) and $f(x)\ge f(\bar{x})+\xi_2^t(x-\bar{x})\ \ \forall x\in \mathbf{R}^n$ (Eq. 2).

(b) (5%) Following (a), what is the inequality of the sum of (Eq. 1) multiplied by λ and (Eq. 2) multiplied by $1-\lambda$?

(c) (10%) Prove that $\partial f(\bar{x})$ is a convex set. (Hint: Show that $\lambda\xi_1+(1-\lambda)\xi_2 \in \partial f(\bar{x})$.)

(d) (10%) Let $f:\mathbf{R}\to\mathbf{R}$ be defined by $f(x)=\max\left\{x^2,(2x+3)\right\}$. Please write the mathematical expression of the sets $\partial f(0)$ and $\partial f(3)$?

5. (20%) Consider the following optimization problem.

$$\begin{aligned}&\min f(x)\\&s.t.\\&g_i(x)\le 0\ \text{ for } i=1,\ldots,m\\&x\in X\end{aligned}$$

The Karush-Kuhn-Tucker (KKT) necessary optimality condition for the above problem is

$$\begin{aligned}&\nabla f(\bar{x})+\sum_{i=1}^{m}u_i\nabla g_i(\bar{x})=0\\&u_i g_i(\bar{x})=0 \quad \text{for } i=1,\ldots,m\\&u_i\ge 0 \qquad\quad \text{for } i=1,\ldots,m\end{aligned}$$

where $\bar{x}$ is a locally optimal solution. The conditions $\nabla f(\bar{x})+\sum_{i=1}^{m}u_i\nabla g_i(\bar{x})=0$ and $u_i\ge 0$ for $i=1,\ldots,m$ are called dual feasibility conditions and the conditions $u_i g_i(\bar{x})=0$ for $i=1,\ldots,m$ are called complementary slackness conditions. Consider the following optimization problem.

$$\begin{aligned}&\text{Minimize } (x_1-3)^2+(x_2-2)^2\\&\text{s.t. } x_1^2+x_2^2\le 5\\&\quad x_1+2x_2\le 4\\&\quad x_1, x_2\ge 0\end{aligned}$$

(a) (10%) Please write the dual feasibility and complementary conditions for this problem.

(b) (5%) Determine an optimal solution **graphically** on the x_1 and x_2 plane. (The horizontal axis is x_1 and the vertical axis is x_2. Hint: The considered optimization problem is to find out the minimum distance from the point (3, 2) to the feasible region.)

(c) (5%) The KKT dual feasibility conditions $-\nabla f(\bar{x})=\sum_{i\in I}u_i\nabla g_i(\bar{x})$ and $u_i\ge 0$ for $i\in I$, where I is the index set of binding constraints at $\bar{x}$, can be geometrically interpreted as that $-\nabla f(\bar{x})$ is expressed as a liner sum of $\nabla g_i(\bar{x})$ for $i\in I$ with nonnegative scaling factors only; i.e. they 'cancel' out each other. Please verify that the solution obtained from (b) satisfies the KKT dual feasibility conditions geometrically. (Please draw the vector $-\nabla f(\bar{x})$ and $\nabla g_i(\bar{x})$ for $i\in I$ with corresponding $u_i\ge 0$ on the graph and conclude the geometrical interpretation.)

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