# 國立臺灣大學 105 學年度碩士班招生考試試題

科目：微積分(D)
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(1) (20 pts) Note that $F(x) = \exp(-\exp(-x))$.

**(a)** (8 pts) Determine $\lim_{x\to\infty} F(x)$ and $\lim_{x\to-\infty} F(x)$.

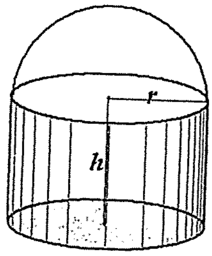
**(b)** (12 pts) Find $a_n$ such that $F(a_n) = 1-n^{-1}$ and determine $\alpha$ such that $\lim_{n\to\infty}[\exp(a_n)]/n^\alpha$ converges to a nonzero constant.

(2) (10%) Suppose $f(x)$ has a continuous third derivative on $[-1,1]$. Determine whether the following series

$$\sum_{n=1}^{\infty}\left[nf\left(\frac{1}{n}\right) - nf\left(-\frac{1}{n}\right) - 2f^{(1)}(0)\right]$$

converges. Please show your work.

(3) (15%) You need to construct a tank consisting of a right circular cylinder with height $h$ and radius $r$ topped with a hemispherical top, and with a flat base, as shown in the figure. If the material for the hemispherical top costs \$20/$m^2$, and the material for the cylindrical side costs \$8/$m^2$, and material for the circular bottom costs \$5/$m^2$, find the value of $r$ and $h$ that minimize the cost of the materials for this tank, assuming that the volume must be $20\pi m^3$. Then what does the ratio $h/diameter$ equal?




(4) (20 pts) Consider $B_p(r)$ which is the ball, centered at the origin, with radius $r$ in $R^p$. Or, $B_p(r) = \{(x_1, \cdots, x_p) : \sum_{i=1}^p x_i^2 \le r^2\}$.

**(a)** (4 pts) Consider the intersection of $B_p$ with the plane $x_1 = r/2$. Is it still a ball? Give reason to support your answer. If it is still a ball, give the coordinate of the center of the ball and its radius.

**(b)** (6 pts) Let $V_p(r)$ denote the volume of the $p$-ball with radius $r$. Show that, for $p \ge 3$,

$$V_p(r) = \frac{2\pi r^2}{p} V_{p-2}(r).$$

**(c)** (5 pts) Use the recurrence relationgiven in (b) to obtain a formula for $V_p(r)$.

**(d)** (5 pts) Find the following limit.

$$\lim_{p\to\infty} \frac{\ln(\ln V_p(1))}{p \ln p}.$$

(5) (10 pts) Show that

$$\int_0^1 \frac{1+x^{30}}{1+x^{60}}dx = 1 + \frac{C}{31}$$

for some $C$ such that $0 < C < 1$.

(6) (10 pts) Use Green's Theorem to evaluate

$$\int\int_E (x^2+y^2)dxdy,$$

where $E$ is the solid ellipse $x^2/a^2 + y^2/b^2 \le 1$.

(7) (15%) The approximation from *Simpsons Rule* for $\int_a^b f(x)dx$ is

$$S_{[a,b]}f = \left[\frac{2}{3}f\left(\frac{a+b}{2}\right) + \frac{1}{3}\left(\frac{f(a)+f(b)}{2}\right)\right](b-a).$$

If $f$ has continuous derivatives up to order three, prove that

$$\left|\int_a^b f(x)dx - S_{[a,b]}f\right| \le C(b-a)^4 \max_{[a,b]} |f^{(3)}(x)|,$$

where $C$ does not depend on $f$.

試題隨卷繳回