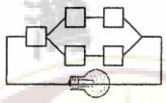
I. Multiple Choice (single answer): 50%

※ 注意：請用 2B 鉛筆作答於答案卡，並先詳閱答案卡上之「畫記說明」。

1. Which of the following statements is NOT true? (a) For any event A, P[A]>0. (b) Let S be the sample space, P[S]=1. (c) If A and B are mutually exclusive, P[A∪B]=P[A] + P[B]. (d) None of them above.

2. Peter throws three dices, each with 6 faces (e.g., 1~6). What is the probability that the sum of the three dices = 8? (a) 18/216 (b) 21/216 (c) 25/216 (d) 27/216

3. What is the probability that the circuit below will be operational (i.e., the light bulb is ON) if the failure probability of each of the 5 components is 0.5? (a) 3/32 (b) 5/32 (c) 7/32 (d) 9/32?



4. Let X_i be a discrete random variable with a probability mass function (PMF) = $p_{X_i}(x) = p(1-p)^{x-1}$, x=1, 2, ... Assume there are 3 such identical and independently (*iid*) random variables. What is the PMF of the sum of these 3 *iid* random variables, say Y= $X_1 + X_2 + X_3$? (a) $p_Y(y) = \frac{y!}{2!(y-2)!} p^3 (1-p)^{y-3}$ (b) $p_Y(y) = \frac{(y-1)!}{2!(y-3)!} p^2 (1-p)^{y-2}$ (c) $p_Y(y) = \frac{y!}{3!(y-3)!} p^3 (1-p)^{y-3}$, (d) $p_Y(y) = \frac{(y-1)!}{2!(y-3)!} p^3 (1-p)^{y-3}$

5. Given Y from Problem 4, what is the expected value of Y given p=0.5? (a) 6 (b) 4 (c) 3/2 (d) 2

6. Let X be the arrival time of the first bus, as measured in units of minutes after 6 AM of a given day. Let Y be the arrival time of the second bus. X and Y are continuous random variables with joint PDF $f_{X,Y}(x,y) = 0.25e^{-0.5y}$, $0 \le x < y$. If the first bus arrived 5 minutes ago, how long on average does the passenger in the bus stop have to wait until the second bus arrives? (a) 5 mins (b) 4 mins (c) 2 mins (d) 1 min

7. Which of the following statements about Gaussian random variables with non-zero mean, X and Y, is/are NOT true? (a) If P[X∩Y]=P[X]*P[Y], E[(X-E[x])(Y-E[Y])]=0. (b) If E[XY]= E[x]E[Y], P[X∩Y]=P[X]*P[Y]. (c) If P[X∩Y]=P[X]*P[Y], Var[X+Y]=Var[X]+Var[Y]. (d) None of them

8. A total of 7 guests go to a party in a rainy night, each bringing an umbrella. What is the average number of guests that get their umbrellas back if everyone randomly picks an umbrella from the bucket when they leave? (a) 1 (b) 7/8 (c) 7/4 (d) 7/2

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9. A and B throw a fair dice with 6 faces. If A's throw is larger than B's, B has to give A 1 dollar. If A's throw is smaller than B's, A has to give B 1 dollar. Otherwise, they keep their money. This game continues until one of them has no money. What is the average number of total throws from A when the game is over given that originally A has 1 dollar and B has 2 dollars to begin with? (a) 5/2 (b) 13/6 (c) 12/5 (d) 3

10. Given the same setting in Problem 9, what is the probability that A will win the game eventually? (a) 1/2 (b) 1/3 (c) 1/4 (d) 2/5

2 Non-multiple choice problems (15%)

※ 注意：請於試卷內之「非選擇題作答區」作答，並應註明作答之題號。

1. (8%) Suppose A and B are $n \times n$ matrices.
   (a) (3%) Show that $AB = BA$ is not necessarily correct for all n.
   (b) (5%) Suppose $AB = I_n$. Show that $BA = I_n$ (i.e., show that A is invertible.)
2. (7%) Consider the following system of linear equations.

$$\begin{cases} x_1 + x_2 + 2x_3 + x_4 = 2 \\ 2x_1 + x_2 + x_3 - x_4 = 3 \\ x_1 - 2x_2 - x_3 - 2x_4 = 5 \end{cases}$$

   (a) (1%) Formulate the problem into a matrix equation $A\mathbf{x} = \mathbf{b}$ where $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$. Write the matrices A and $\mathbf{b}$ explicitly.
   (b) (3%) Perform Gaussian Elimination on the augmented matrix $[\ A \mid \mathbf{b}\ ]$ and obtain the reduced row echelon form $[\ R \mid \mathbf{c}\ ]$.
   (c) (3%) Find the solution set of the system of linear equations.

3 Multiple Choices (Multiple Answers) (35%)

※ 注意：請於試卷內之「非選擇題作答區」作答，並應註明作答之題號。

3. (5%) Which of the following is a linear transformation?
   (A) $T : \mathcal{R} \to \mathcal{R}$ where $T(x) = 2x + 1$ for any $x \in \mathcal{R}$.
   (B) $T : \mathcal{R}^2 \to \mathcal{R}^2$ where $T(\mathbf{x}) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \mathbf{x}$ for any vector $\mathbf{x} \in \mathcal{R}^2$.
   (C) $T : \mathcal{P} \to \mathcal{P}$ where $T(f(x)) = f(x)(x+1)$ for any polynomial $f(x) \in \mathcal{P}$.
   (D) $T : \mathcal{C}(\mathcal{R}) \to \mathcal{C}(\mathcal{R})$ where $T(f(x)) = 3f''(x) + f'(x) + f(x)$ for any differentiable function $f(x) \in \mathcal{C}(\mathcal{R})$.
   (E) None of the above.
4. (5%) Which of the following sets is an orthonormal basis for the designated vector space?
   (A) $\left\{ \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \end{bmatrix} \right\}$, for $\mathcal{R}^2$ with $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \mathbf{y}$.
   (B) $\{\cos x, \sin x\}$, for $\mathcal{C}[0, 2\pi]$ with $\langle f, g \rangle = \int_0^{2\pi} f(x)g(x)dx$.
   (C) $\{1, x, x^2\}$ for $\mathcal{P}_3$ with $\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)dx$.
   (D) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ for $\mathcal{R}^3$ with $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T M \mathbf{y}$ where $M = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$
   (E) None of the above.

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5. (5%) Which of the following is true?

   (A) For any matrices A and B, if $AB = I$, then $BA = I$.

   (B) For any $m \times n$ matrices A and B, if rank$A$ = rank$B$, then rank$A$ = rank$(A + B)$.

   (C) For any $m \times n$ matrix A, rank$A = m$ if and only if $A\mathbf{x} = \mathbf{b}$ has at least a solution for all $\mathbf{b} \in \mathcal{R}^m$.

   (D) For any $m \times n$ matrix A, rank$A = n$ if and only if $A\mathbf{x} = \mathbf{0}$ has only one solution.

   (E) None of the above.

6. (5%) Which of the following is true?

   (A) Let W be a subspace of vector space V, and let $\mathcal{B}$ be a basis for W. Then there exists a linear independent subset of V, $\mathcal{B}_2$, such that $\mathcal{B} \bigcup \mathcal{B}_2$ is a basis for V.

   (B) Let V be a vector space and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_n\}$ be a basis for V. Then there exists a vector $\mathbf{v} \in V$ such that $\mathbf{v}$ can be expressed as a linear combination of $\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_n$ with infinitely many different sets of coefficients.

   (C) Suppose a subset of V, $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ is a generating set for the vector space V, then there exists a unique subset of $\mathcal{S}, \mathcal{B} \subset \mathcal{S}$, such that $\mathcal{B}$ is a basis for V.

   (D) Let V be a vector space and let the dimension of V be n. Let S be a subset of V containing $n + 1$ vectors. Then S must be a linearly dependent generating set for V

   (E) None of the above.

7. (5%) Let A be an $n \times n$ matrix, then which of the following is true?

   (A) Suppose $A\mathbf{v} = \lambda\mathbf{v}$ where λ is a nonzero scalar. Then $\mathbf{v}$ must be an eigenvector of A and λ must be an eigenvalue of A.

   (B) If both $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of A, then $\mathbf{v}_1 + \mathbf{v}_2$ is also an eigenvector of A.

   (C) If A contains n distinct eigenvalues, then A is diagonalizable.

   (D) If A has n linearly independent eigenvectors, then A is diagonalizable.

   (E) None of the above.

8. (5%) Let A be an $n \times n$ real-valued matrix, then which of the following is true?

   (A) If A is similar to the identity matrix I_n, then $\det A = 1$.

   (B) If $A^T = A$, then all eigenvalues of A are real numbers.

   (C) If $\det A > 0$, then rank$A > n - 1$.

   (D) If A is diagonalizable, then $A + A^2 + A^3$ is also diagonalizable.

   (E) None of the above.

9. (5%) Let $(V, \mathcal{R}, \text{“}+\text{”}, \text{“}\cdot\text{”})$ be a vector space. Prove that $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, $\mathbf{u} + \mathbf{v} = \mathbf{u} + \mathbf{w}$ implies $\mathbf{v} = \mathbf{w}$. Which of the following axioms of vector spaces was NOT necessarily used in the proof?

   (A) For all $\mathbf{u}, \mathbf{w}, \mathbf{v} \in V$, $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.

   (B) For all $\mathbf{u}, \mathbf{v} \in V$, $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

   (C) For all $\mathbf{u} \in V$, there exists $\mathbf{v} \in V$ such that $\mathbf{u} + \mathbf{v} = 0$.

   (D) For all $\mathbf{u} \in V$, $1 \cdot \mathbf{u} = \mathbf{u}$.

   (E) All of the above are needed in the proof.

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