1. To answer:

(*a*) (5%) Evaluate $\lim_{x\to 1}\dfrac{1+\cos \pi x}{x^2-2x+1}$

(*b*) (5%) If $\phi(\alpha)=\int_{\alpha}^{\alpha^2}\dfrac{\sin \alpha x}{x}dx$, find $\phi'(\alpha)$ where $\alpha\neq 0$.

(*c*) (5%) Test for convergence: $\sum_{n=1}^{\infty}\dfrac{1}{n^p}$, p = constant.

(*d*) (5%) Determine whether $\int_{-1}^{5}\dfrac{dx}{(x-1)^3}$ converges. (*i*) in the usual sense, (*ii*) in the Cauchy principal value sense.

2. (10%) Solve $y''+2y'+2y=-10xe^x+5\sin x$. [hint: general solution using undetermined coefficients]

3. (10%) Let x be a random variable with a finite expected value, $\mu=E(x)$. If $\varphi(x)$ is a twice differentiable convex function, then to show: $E[\varphi(x)]\geq\varphi[E(x)]$.

4. (10%) If $f(x)$ and $g(x)$ are continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b). To show: there exists a point c, $a<c<b$, such that $[f(b)-f(a)]g'(c)=[g(b)-g(a)]f'(c)$.

5. (12%) A telephone company is planning to introduce two new types of executive communication systems that it hopes to sell to its commercial customers. It is estimated that if the first type of system is priced at x hundred dollars per system and the second type at y hundred dollars per system, approximately $40-8x+5y$ customers will buy the first type and $50+9x-7y$ customers will buy the second type. If the cost of manufacturing the first type is \$1,000 per system and the cost of manufacturing the second type is \$2,000 per system, how should the telephone company price the systems to generate the largest possible profit?

(a) (4%) Derive the objective function of the total profit and calculate the first-order partial derivatives with respect to x and y, respectively.

(b) (4%) Solve the optimal values of x and y.

(c) (4%) Examine the second-order partial derivatives to verify the optimal values of x and y can generate the largest total profit.

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6. (12%) Consider a variable x which follows a Poisson distribution with the intensity of events occurring to be λ. The probability that there are exactly k events occurring is

$$\text{prob}(x = k) = \frac{\lambda^k e^{-\lambda}}{k!}.$$

Moreover, the intensity parameter, λ, is a nonnegative variable and follows a gamma distribution, i.e., $\lambda \sim \text{Gamma}(\alpha, \beta)$, where α and β are positive real numbers. Consequently, the probability density function for λ is as follows.

$$f(\lambda) = \beta^\alpha \frac{1}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda},$$

where $\Gamma(Z)$ is the gamma function and defined as

$$\Gamma(Z) = \int_0^\infty e^{-t} t^{Z-1} dt,$$

if Z is a complex number with a positive real part.

For the above Gamma-Poisson mixture distribution, the probability that there are exactly k events occurring for the variable x can be expressed in the following form.

$$\text{prob}(x = k) = \frac{\Gamma(a)}{\Gamma(b)k!} (c)^a (d)^k.$$

Solve a, b, c, and d as functions of α, β, λ, and k.

7. (26%) The Gaussian Quadrature (GQ) method provides a numerical way to generate the approximation of the definite integral of any function $f(x)$ as follows.

$$\int_{-1}^{1} f(x)dx \approx \sum_{i=1}^{n} w_i f(x_i).$$

Different from the traditional numerical integration methods, such as the rectangular rule, trapezoid rule, or Simpson's rule, the GQ method gives users the freedom to choose not only the location of the abscissas, x_i, but also the weighting coefficients, w_i, such that GQ can generate the exact integral results of the polynomial functions with degree $2n - 1$ or less in $[-1,1]$. Therefore, x_i and w_i can be solved based on the following $2n$ equations.

$$\int_{-1}^{1} x^l \, dx = \sum_{i=1}^{n} w_i x_i^l \text{ for } l = 0,1,\ldots,2n-1.$$

For the following problems, consider the case of $n = 2$.

(a) (12%) Solve w_1, w_2, x_1, and x_2.

(b) (6%) Verify that for any cubic polynomial function, $f(x) = ax^3 + bx^2 + cx + d$, the GQ method can generate the exact integral results.

(c) (8%) Suppose $f(x) = e^{-\frac{1}{9}x^2 + \log_3 |x|}$ is not integrable. Apply the GQ method to estimate

$$\int_{-3+2\sqrt{3}}^{3+2\sqrt{3}} f(x)\, dx.$$

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