1. (15%) Consider the following wave equation, i.e.,

$$\frac{\partial^2\theta}{\partial\tau^2}=\frac{\partial^2\theta}{\partial\xi^2}+\xi^2, \quad (0<\xi<h,\ 0<\tau) \tag{1}$$

subjected to the boundary conditions of $\theta(0,\tau)=\theta(h,\tau)=0$ and the initial conditions of $\partial_\tau\theta(\xi,0)=\theta(\xi,0)=0$ (note: $\partial_\tau=\partial/\partial\tau$). Solve Eq. (1) through the following steps:

(a) (3%) Which one of the following transformations or change of variables is most likely to help you solve Eq. (1)? Let:

(A) $\theta(\xi,\tau)=\Xi(\xi)\mathrm{T}(\tau)$,

(B) $\theta(\xi,\tau)=\Xi(\xi)\mathrm{T}(\tau)+\Psi(\xi)$.

(b) (6%) After transforming the variables in Eq. (1) using one of the transformations listed in (a), what are the governing differential equations as well as the new boundary and/or initial conditions for $\Xi(\xi)$ and $\mathrm{T}(\tau)$?

(c) (6 %) Find the solution to the original Eq. (1).

2. (15%)

(a) (3%) Let $F(s)$ denote the Laplace transform of $f(t)$, i.e., $\mathcal{L}[f(t)]=F(s)$. What is the Laplace transform of $\frac{\partial f}{\partial t}$ in terms of $F(s)$, s, and the initial condition of $f(t)$?

(b) (3%) Take the Laplace transform with respect to the t-coordinate of the following partial differential equation along with its associated initial and boundary conditions, i.e.,

$$\frac{\partial\varphi}{\partial t}=\alpha\frac{\partial^2\varphi}{\partial x^2}, \tag{2}$$

with $\varphi(x,0)=\varphi_0$, $\varphi(0,t)=0$, and $\varphi(\infty,t)=\varphi_0$ for all $t>0$ in the domain of $0<x<\infty$, where φ_0 is a constant.

(c) (3%) Solve the differential equation resulting from (b).

(d) (3%) Which one of the following is correct and helpful to solving this problem?

(A) $\mathcal{L}\left[\mathrm{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right)\right]=\frac{1}{s}\exp\left(-x\sqrt{\frac{s}{\alpha}}\right)$,

(B) $\mathcal{L}\left[J_0\left(\frac{x^2}{\alpha}t\right)\right]=\frac{1}{s}\exp\left(-x\sqrt{\frac{s}{\alpha}}\right)$,

(C) $\mathcal{L}\left[\cosh\left(\frac{x^2}{\alpha}t\right)\right]=\frac{1}{s}\exp\left(-x\sqrt{\frac{s}{\alpha}}\right)$,

(D) $\mathcal{L}\left[\sin\left(\frac{x^2}{\alpha}t\right)\right]=\frac{1}{s}\exp\left(-x\sqrt{\frac{s}{\alpha}}\right)$

(e) (3%) Find the solution to Eq. (2).

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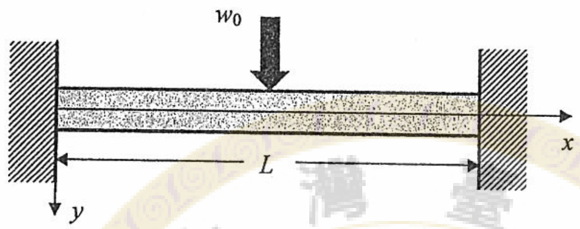
3. (15%) A uniform beam of Length L carries a concentrated load w_0 at $x = L/2$. The beam is clamped at both ends. **<u>Use the Laplace transform</u>** to determine the deflection $y(x)$ from

$$EI\frac{d^4y}{dx^4} = w_0\delta\left(x - \frac{L}{2}\right)$$

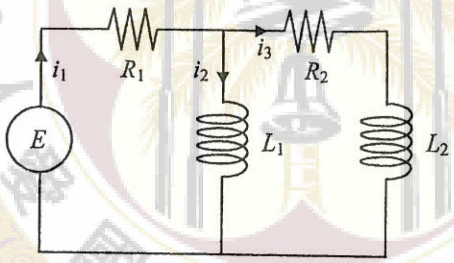
where $y(0)=0$, $y'(0)=0$, $y(L)=0$, $y'(L)=0$, and $\delta\left(x-\frac{L}{2}\right)$ is the Dirac delta function.



4. (15%) The system of differential equations for the currents $i_2(t)$ and $i_3(t)$ in the electrical network is

$$\frac{d}{dt}\begin{bmatrix} i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} -R_1/L_1 & -R_1/L_1 \\ -R_1/L_2 & -(R_1+R_2)/L_2 \end{bmatrix}\begin{bmatrix} i_2 \\ i_3 \end{bmatrix} + \begin{bmatrix} E/L_1 \\ E/L_2 \end{bmatrix}$$

Solve the system if $R_1 = 2\ \Omega$, $R_2 = 3\ \Omega$, $L_1 = 1$ h, $L_2 = 1$ h, $E = 60$ V, $i_2(0) = 0$, and $i_3(0) = 0$.



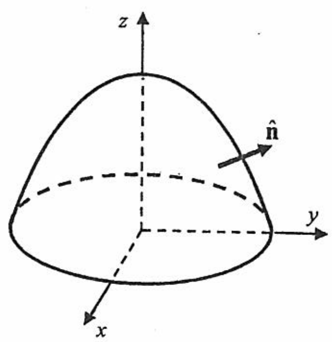
5. (10%) Solve the following differential equation if it is known that two linearly independent solutions of the associated homogeneous differential equation are t and t^2.

$$t^2\frac{d^2N}{dt^2} - 2t\frac{dN}{dt} + 2N = t(\ln t)$$

6. (8%) Evaluate the surface integral

$$J = \int_A \hat{\mathbf{n}}\cdot\nabla\times\mathbf{v}\,da$$

where $\mathbf{v} = x\hat{\mathbf{j}} - (z+1)\hat{\mathbf{k}}$, A is the surface $z = 4 - 4x^2 - 4y^2$ between $z = 0$ and $z = 4$, and $\hat{\mathbf{n}}$ is the unit normal whose direction is shown in the figure.



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7. (7%) Determine the Laurent expansion of

$$f(z)=\frac{1}{z(2-z)}$$

about $z=0$ in $|z|>2$.

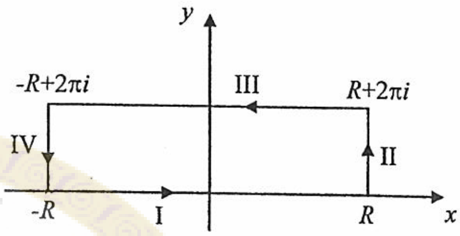
8. (15%) Suppose that $0<a<1$. In order to evaluate

$$I=\int_{-\infty}^{\infty}\frac{e^{ax}}{1+e^{x}}dx,$$

we first consider the contour integral

$$J=\oint_{\Gamma}\frac{e^{az}}{1+e^{z}}dz.$$

where Γ, as shown at right, is the counterclockwise rectangular with vertices at $-R$, R, $R+2\pi i$, and $-R+2\pi i$.



(a) Find and classify all singular points of $\frac{e^{az}}{1+e^{z}}$ inside Γ.

(b) Determine $J=\oint_{\Gamma}\frac{e^{az}}{1+e^{z}}dz$.

(c) Given that both line integrals along the two vertical segments (II and IV as shown in the figure) tend to zero as $R\to\infty$ (i.e., $\lim_{R\to\infty}\left|\int_{\mathrm{II}}\frac{e^{az}}{1+e^{z}}dz\right|=0$ and $\lim_{R\to\infty}\left|\int_{\mathrm{IV}}\frac{e^{az}}{1+e^{z}}dz\right|=0$), determine $I=\int_{-\infty}^{\infty}\frac{e^{ax}}{1+e^{x}}dx$ using the above results.

試題隨卷繳回