※選擇題部分請作答於電腦答案卡上，並且限用 **2B** 鉛筆。

第一大題：多重選擇題
＊本大題各題答案應作答於答案卡，否則不予計分。
＊每題有一個以上正確選項。每一選項答錯(應選而未選或不應選而選)扣2.5分。
＊每題未作答或答錯兩個以上選項者，該題以零分計。

1. (5%) Consider the following statements about two random variables X and Y:

   i. If X and Y are independent, the correlation of X and Y must be 0.
   ii. If X and Y are uncorrelated, the correlation of X and Y must be 0.
   iii. If X and Y are independent, $E[XY]$ must equal to $E[X] \cdot E[Y]$.
   iv. If X and Y are correlated, $E[X+Y]$ may not equal to $E[X]+E[Y]$.

   Which of the statements above is(are) **TRUE**?
   (A) i (B) ii (C) iii (D) iv (E) None of the above.

2. (5%) Random variables X_1, X_2, X_3, X_4 have the joint PDF:

$$f_{X_1,X_2,X_3,X_4}(x_1,x_2,x_3,x_4) = \begin{cases} C & 0 \le x_1 \le x_3 \le 1, 0 \le x_2 \le x_4 \le 2, \\ 0 & otherwise, \end{cases}$$

   where C is a constant. Consider the following statements:

   i. The value of the constant C is 2.
   ii. The marginal PDF of X_1 is $f_{X_1}(x_1) = \begin{cases} 2x_1 & 0 \le x_1 \le 1, \\ 0 & otherwise. \end{cases}$
   iii. Random variables X_1, X_2, X_3, X_4 are independent.
   iv. Random vectors $[X_1, X_3]$ and $[X_2, X_4]$ are independent.

   Which of the statements above is(are) **TRUE**?
   (A) i (B) ii (C) iii (D) iv (E) None of the above.

3. (5%) Consider a random variable X with the PDF:

$$f_X(x) = 0.15[u(x) - u(x-2)] + 0.1e^{-\frac{x}{3}}u(x) + \frac{C'}{\sqrt{\pi}}e^{-\frac{(x-2)^2}{4}},$$

   where $u(x)$ is the unit step function and C' is a constant. Consider the following statements:

   i. The value of the constant C' is 0.1.
   ii. $E[X] = 2$.
   iii. The variance of X is 3.2.
   iv. $P\{X \le 2\} = 0.8 - 0.3e^{-\frac{2}{3}}$.

   Which of the statements above is(are) **TRUE**?
   (A) i (B) ii (C) iii (D) iv (E) None of the above.

4. (5%) Random variables W, X, Y, Z have the following relations:

$$\begin{aligned} W &= X - 0.5, \\ X &= 0.9Y + 0.2, \\ Y &= -0.5Z + 0.3. \end{aligned}$$

   Consider the following statements about the correlation coefficients:

   i. $\rho_{W,X} = 1$.
   ii. $\rho_{X,Y} = 0.9$.
   iii. $\rho_{Y,Z} = -0.5$.
   iv. $\rho_{W,Z} = -1$.

   Which of the statements above is(are) **TRUE**?
   (A) i (B) ii (C) iii (D) iv (E) None of the above.

5. (5%) Consider an experiment that produces observations of sample values of a random variable X with unknown yet finite variance $Var[X]$ and mean $E[X]$. The observed sample value of the i-th trial is denoted by X_i. Define $M_n(X) = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $V_n(X) = \frac{1}{n}\sum_{i=1}^{n}(X_i - M_n(X))^2$. Consider the following statements:

   i. $M_n(X)$ is an unbiased estimate of $E[X]$.
   ii. $V_n(X)$ is an unbiased estimate of $Var[X]$.
   iii. $V_n(X)$ is an asymptotically unbiased estimate of $Var[X]$.
   iv. $\{M_n(X)\}$ is a sequence of consistent estimates of $E[X]$.

   Which of the statements above is(are) TRUE?
   (A) i (B) ii (C) iii (D) iv (E) None of the above.

6. (5%) Which of the following subsets of $\mathbb{R}^3$ is(are) linearly independent?

   (A) $\{(1,0,0),(0,2,0),(0,0,1)\}$
   (B) $\{(1,2,3),(0,-1,0)\}$
   (C) $\{(1,0,1),(1,1,0),(0,0,1),(0,1,1)\}$
   (D) $\{(1,2,1),(2,4,-2)\}$
   (E) None of the above.

7. (5%) Which of the following subsets of $\mathbb{R}^3$ is(are) an orthogonal basis(bases) thereof?

   (A) $\{(1,0,0),(0,2,0),(0,0,1)\}$
   (B) $\{(1,0,0),(0,1,0),(0,0,0)\}$
   (C) $\{(3,4,0),(-4,3,0),(0,0,7)\}$
   (D) $\{(1,1,1),(1,-1,0),(1,1,-2)\}$
   (E) None of the above.

8. (5%) Which of the following is(are) a linear transformation(s)?

   (A) $T:\mathbb{R}^2 \to \mathbb{R}^3, T(x,y) = (x-y,x,y)$
   (B) $T:\mathbb{R}^2 \to \mathbb{R}^3, T(x,y) = (0,x,y^2)$
   (C) $T:\mathbb{R}^3 \to \mathbb{R}^3, T(x,y,z) = (x+z,y-z,y-x)$
   (D) $T:\mathbb{R}^3 \to \mathbb{R}^3, T(x,y,z) = (x+1,x-y,z)$
   (E) None of the above.

9. (5%) Which of the following is(are) an inner product(s) defined on $\mathbb{R}^4$? Note that $\mathbf{u} = (u_1,u_2,u_3,u_4)$ and $\mathbf{v} = (v_1,v_2,v_3,v_4)$ are any given vectors in $\mathbb{R}^4$.

   (A) $\langle \mathbf{u},\mathbf{v}\rangle = u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4$.
   (B) $\langle \mathbf{u},\mathbf{v}\rangle = u_1v_1 + 2u_2v_2 + 3u_3v_3 + 4u_4v_4$.
   (C) $\langle \mathbf{u},\mathbf{v}\rangle = u_1v_1 - u_2v_2 + u_3v_3 + u_4v_4$.
   (D) $\langle \mathbf{u},\mathbf{v}\rangle = u_1v_1 + u_2v_2 + u_4v_4$.
   (E) None of the above.

10. (5%) Consider an $m \times n$ matrix A. Which of the following is(are) TRUE?

   (A) $\text{rank}(A) + \text{nullity}(A) = m$.
   (B) The dimension of the row space of A equals to that of the column space of A.
   (C) $1 \leq \text{rank}(A) \leq \min(n,m)$.
   (D) AA^T is always invertible.
   (E) None of the above.

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11. (5%) Which of the following is(are) an eigenvector(s) of $A = \begin{bmatrix} 0 & -2 & 3 \\ -2 & -9 & 12 \\ -2 & -8 & 11 \end{bmatrix}$?

(A) $\mathrm{x} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$; (B) $\mathrm{x} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$; (C) $\mathrm{x} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$; (D) $\mathrm{x} = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$;
(E) None of the above.

第二大題：非選擇題
＊本大題應作答於試卷，作答於試題紙上不予計分。

1. (10%) Please derive the following MGFs:

   (a) (5%) If X has the PMF:

   $$P_X(x) = \begin{cases} \alpha^x e^{-\alpha}/x! & x = 0, 1, 2, \ldots \\ 0 & otherwise. \end{cases}$$

   Please derive its MGF $\phi_X(s) = E[e^{sX}]$. You have to give detailed derivation to get full credits.

   (b) (5%) If X has the PDF:

   $$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-1)^2}{2}}$$

   Please derive its MGF $\phi_X(s) = E[e^{sX}]$. You have to give detailed derivation to get full credits.

2. (15%) A mathematician decides to reconstruct the whole probability theory based on a new set of probability axioms. The new set of probability axioms has three axioms. Two of them are the same as the axioms that we have been used in the original probability theory. The only different one is the axiom that states $P\{S\} = 3$, which is different from the axiom of $P\{S\} = 1$ that we have commonly used. This means that in the new set of probability axioms, probability now summed up to 3, not 1. Based on this new set of probability axioms, the new probability theory is constructed, which is quite different from the original probability theory we have now. However, the same definition of expectation is still used for the new probability theory, i.e.

   $$E[X] = \sum_{x=-\infty}^{\infty} x P\{X = x\},$$

   where $P\{X = x\}$ denotes the probability of $X = x$. Answer the following questions:

   (a) (5%) Consider the experiment of flipping a coin n times. Let random variable X denote the number of heads we see in the experiment. Assume that the coin is a fair coin. According to the new set of probability axioms, what should the probability of $X = x$ be in the new probability theory? Clearly explain why your answer is correct in detail.

   (b) (5%) Does the original law of large numbers (LLN) still hold in the new probability theory? Clearly explain why yes or why not in detail.

   (c) (5%) Does the definition of conditional probability $P\{A|B\} = \frac{P\{A \cap B\}}{P\{B\}}$ still work fine in the new probability theory? Clearly explain why yes or why not in detail.

3. (5%) Let V be a vector space over a field F with addition "+" and scalar multiplication "·". Prove that $c \cdot \mathbf{0} = \mathbf{0}$ for any $c \in F$, where $\mathbf{0}$ is the zero vector in V. Use only axioms of vector spaces (as well as those of fields).

4. (5%) Let W be a subspace of V. Suppose $B = \{\mathrm{v}_1, \mathrm{v}_2, \cdots, \mathrm{v}_m\} \subset V$ is a basis for W. Prove that any subset S of W containing more than m vectors is linearly dependent.
   (Hint: Use the fact that a homogeneous system of linear equations has an infinite number of solutions if it has fewer equations than variables.)

5. (5%)

(a) (3%) Apply Gram-Schmidt orthonormalization process on the subspace of $\mathbb{R}^4$ defined as

$$W = \text{span}\left\{ \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T, \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}^T, \begin{bmatrix} 2 & 0 & 1 & -1 \end{bmatrix}^T \right\}$$

and give an orthonormal basis for W.

(b) (2%) Find an orthonormal basis for $W^{\perp}$, the orthogonal complement of W.

6. (5%) Let $A = \begin{bmatrix} 5 & -4 & 2 \\ 8 & -8 & 6 \\ 4 & -4 & 3 \end{bmatrix}$. Calculate A^{10000}. (*Hint: First find eigenvalues and eigenvectors of A.*)