國立臺灣大學98學年度碩士班招生考試試題

1. (20%) Label the following statements as being true or false. *(No explanation is needed. Each correct answer gets 2% and each wrong answer gets 0%)*:

   (a) If the reduced row echelon form of $[A \;\; b]$ contains a zero row, then $Ax = b$ has infinitely many solutions.

   (b) A function from $\mathcal{R}^n$ to $\mathcal{R}^m$ is uniquely determined by its images of the standard vectors in $\mathcal{R}^n$.

   (c) Let A be an $m \times n$ matrix. Then the rank of A is n if and only if the equation $Ax = b$ has at most one solution for each b in $\mathcal{R}^m$.

   (d) Let A, B and C be *any* matrices such that the product ABC is defined. Then rank $(ABC) \leq$ rank B.

   (e) Let $A = [a_1 \; a_2 \; \ldots \; a_n]$ be a square matrix and b be a linear combination of $a_1, a_2, \ldots, a_n$. Then $\det A = \det[a_1 + b \; a_2 \; a_3 \; \ldots \; a_n]$.

   (f) If V and W are subspaces of $\mathcal{R}^n$ having the same dimension, then $V = W$.

   (g) Every column of A can be *uniquely* expressed as a linear combination of the pivot columns of A.

   (h) Let V be a subspace of $\mathcal{R}^n$ and W be its orthogonal complement. If v is a vector in V and w is a vector in W, then $v \bullet w = 0$.

   (i) Let S be a set containing n linearly independent eigenvectors of an $n \times n$ symmetric matrix. Then S forms an orthogonal basis for $\mathcal{R}^n$.

   (j) A matrix representation of a linear operator on $\mathcal{M}_{m \times n}$ is an $m \times n$ matrix.

2. Let A be the 3×3 matrix defined below. (a) (6%) Find the eigenvalues of A, and (b) (9%) find an orthonormal basis for $\mathcal{R}^3$ consisting of eigenvectors of A.

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}.$$

3. Let T be a linear operator on $\mathcal{P}_2$ defined by

$$T(f(x)) = f(0) + f(x) + f'(x) + f(1)x^2.$$

   (a) (5%) Find $[T]_B$, where B is the standard basis for $\mathcal{P}_2$. (b) (10%) For $f(x) = a_0 + a_1 x + a_2 x^2$, find $T^{-1}(f(x))$.

見背面

4. (4%) Let S={1, 2, 3, 4, 5, 6} and the probability p(s) = 1/6 for all $s \in S$. Given four events E1={ 1, 2, 3}, E2={2, 4}, E3={4}, and E4={2, 3, 4, 5}, which of the following statement are correct
   A. E1 and E3 are mutually independent,
   B. E2 and E3 are mutually independent,
   C. E1 and E4 are mutually independent,
   D. E2 and E4 are mutually independent, or
   E. None of above

5. (4%) A and B are playing a game as follow. First, A and B pick their own numbers uniformly from {1, 2, 3, 4, 5} and {1, 3, 8}, respectively. Assume that A's number (say a1) is smaller than or equal B's number (say b1), then A gets one dollar. Whoever gets one dollar (in this case, A) can pick a new number, say a2, for the next run using his uniform distribution. Whoever does not get one dollar (in this case, B) will use a new number b2=b1-a1 for the next run. The same rule applies to B. (Note that it is possible that both get one dollar in the same run). Let A and B play this game for an infinite number of time, what is the ratio of A's money and B's money at the end of the game?
   A. 3:5
   B. 3:4
   C. 5:3
   D. 5:4
   E. Non of above

6. (4%) Let X1, X2, ... and Xn are independent and identical random variables with a CDF F(x). Let Z=min(X1, X2, ..., Xn). Then the PDF of Z can be represented as
   A. $(dF(x)/dx)^n$
   B. $F(x)^{n-1}dF(x)/dx$
   C. $n*F(x)^{n-1}dF(x)/dx$
   D. $n*[1-F(x)]^{n-1}dF(x)/dx$
   E. None of above

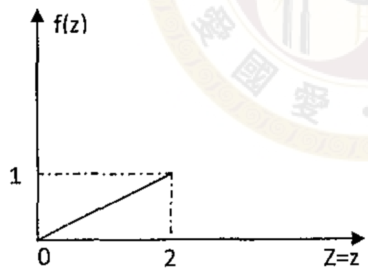
7. (4%) Let $P(Y=y|X=x)=x^y/y!*e^{-x}$ for y=0, 1, 2, ... and X is a zero mean Gaussian random variable with variance = 1. Then E[Y] =
   A. $1/\sqrt{2\pi}$
   B. $1/2\pi$
   C. $\sqrt{2\pi}$
   D. 1/2
   E. None of above

8. (4%) The moment generating function M(s) of a Poisson random variable with mean = α is
   A. $M(s) = \alpha e^{\alpha s)}$
   B. $M(s) = e^{\alpha(e^{s}-1)}$
   C. $M(s) = \alpha!\, e^{\alpha}(1 + e^{s})$
   D. $M(s) = e^{\alpha s+s+1}$
   E. None of above

9. (8%) X is a continuous random variable and is said to be memoryless if Pr(X=t1) = Pr(X =t1+t2|X>t2). Please find the PDF of such an X and show that it is indeed memoryless.

10. X is a random with a PDF such that $\Pr(X>x)=(x/c)^{-k}$ for all $x \geq c$, where c and k are both constants. Please find the mean and variance of X (7%). Please find a random variable X and its PDF such that X has a finite mean but an infinite variance (3%).

11. Let X have a CDF F(x) and Y=F(X). Please find the PDF of Y (4%) and calculate Pr(Y > 0.6) (2%). If we have a uniform random variable generator, please show how to generate a random variable Z such that Z's pdf f(Z=z) is as shown in the figure below (6%).



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