1. Solve the differential equation: (10 %)

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = 0$$

2. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of a given matrix $A=[a_{jk}]$. In each case, prove the proposition. (20%)
   (1) The transpose A^T has the same eigenvalues as A.
   (2) The so-called trace of A, given by trace $A=a_{11}+a_{22}+\ldots+a_{nn}$, is equal to $\lambda_1+\lambda_2+\ldots+\lambda_n$.
   (3) If A is triangular, the elements of the principal diagonal are the eigenvalues of A.
   (4) The matrix A-kI has the eigenvalues $\lambda_1-k, \lambda_2-k, \ldots, \lambda_n-k$.

3. (1) What is the Fourier transform? (5 %)
   (2) What is the Laplace transform? (5 %)
   (3) Please show that: The Laplace transform is a natural result of providing the Fourier transform with a built-in convergence factor. (10 %)

4. Consider the O.D.E.:

$$y'' + 6y' + 34y = u(t)$$
$$y(0) = 0;\ y'(0) = 10$$

   where $u(t)$ is an unit step function

$$u(t) = 1 \text{ for } t > 0;\ u(t) = 0 \text{ for } t < 0.$$

   calculate $y(t)$ using "Laplace transform" methods. (10 %)

5. (1) Describe the Stokes' theorem, and explain clearly the symbols used in your equations. (10 %)

   (2) Let S be the paraboloid: $z = 9 - x^2 - y^2$, which define over the disk in the xy-plane with radius 3. Please verify Stokes' theorem for the vector field (10 %)

$$\vec{F} = (2z - y)\vec{\imath} + (x + z)\vec{\jmath} + (3\mathrm{x} - 2\mathrm{y})\vec{k}.$$

6. A body of constant mass m is projected away from the earth in a direction perpendicular to the earth's surface with an initial velocity v_0. Assuming that there is no air resistance, but taking into account the variation of the earth's gravitational field with distance, we propose to find an expression for the velocity during the ensuing motion. The gravitational force acting on the body (that is, its weight) is inversely proportional to the square of the distance from the center of the earth and is given by

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$$w(x) = -\frac{mgR^2}{(R+x)^2}$$

where m is the mass, g is the acceleration due to gravity at sea level, R is the radius of the earth and x is the distance from the sea level. (15 %)

(a) Use Newton's second law to find the equation of motion.

(b) Show that: $\frac{dv}{dt} = v\frac{dv}{dx}$, and to find an ODE for $v(x)$.

(c) Find the initial velocity v_0 (when $x = 0$ at $t = 0$, the velocity of the body is known to be $v(0) = v_0$), that is required to lift the body to a given maximum altitude ξ above the surface of the earth. The maximum altitude is where the body stops ascending and starts falling back to the earth.

7. Find the Taylor series of $\log(1 + x)$. (5 %)

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